

MORSE THEORY FOR THE TRAVEL TIME BRACHISTOCHRONES IN STATIONARY SPACETIMES

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ABSTRACT. The travel time brachistochrone curves in a general relativistic framework are timelike curves, satisfying a suitable conservation law with respect to an observer field, that are stationary points of the travel time functional. In this paper we develop a global variational theory for brachistochrones joining an event p and the worldline of an observer γ in a stationary spacetime \mathcal{M} . More specifically, using the method of Lagrange multipliers, we compute the first and the second variation of the travel time functional, obtaining two variational principles relating the geometry of the brachistochrones with the geometry of geodesics in a suitable Riemannian structure. We present an extension of the classical Morse Theory for Riemannian geodesics to the case of travel time brachistochrones, and we prove a Morse Index Theorem for brachistochrones. Finally, using techniques from Global Analysis, we prove the Morse relations for the travel time functional and we establish some existence and multiplicity results for brachistochrones.

1. INTRODUCTION: THE GENERAL RELATIVISTIC BRACHISTOCHROME PROBLEM

The classical brachistochrone problem dates back to the end of the seventeenth century, when Johann Bernoulli challenged his contemporaries to solve the following problem.

If in a vertical plane two points A and B are given, then it is required to specify the orbit AMB of the movable point M , along which it, starting from A , and under the influence of its own weight, arrives at B in the shortest possible time. (*Acta Eruditorum*, June 1696)

This problem attracted the attention of many important mathematicians of the time, including Newton, Leibniz, L'Hôpital, and Johann's brother, Jakob Bernoulli. The papers written on the subject may be considered the fundamentals of a new field in mathematics, the *Calculus of Variations*. A beautiful historical exposition of the brachistochrone problem may be found in Reference [27], where the authors' thesis is that the brachistochrone problem also *marks the birth of Optimal Control*.

Still now the classical brachistochrone problem is very popular, and its importance is witnessed by the fact that there is hardly any book on Calculus of Variations that does not use this problem as a takeoff point. The well known solution to the brachistochrone problem is a cycloid, which is the curve described by a point P in a circle that rolls without slipping.

This problem has several generalizations, e.g., the homogeneous gravitational field could be replaced with an arbitrary Newtonian potential, and instead of releasing the particle from rest one could prescribe an arbitrary value for the initial speed, leaving the initial direction of the velocity undetermined.

In modern terminology, the Newtonian brachistochrone problem can be stated as follows. Given a manifold \mathcal{M}_0 endowed with a Riemannian metric g_0 , to be interpreted as the state space, and a smooth function $V : \mathcal{M}_0 \mapsto \mathbb{R}$, representing the gravitational potential, a brachistochrone of energy $E > 0$ between two points x_0 and x_1 of \mathcal{M} is a curve $x : [0, T_x] \mapsto \mathcal{M}$ joining x_0 and x_1 that extremizes the travel time T_x in the space of all curves y joining x_0 and

x_1 and satisfying the conservation of energy law:

$$(1.1) \quad \frac{1}{2} g(\dot{y}, \dot{y}) + V(y) \equiv E$$

(throughout this paper we will consider the motion of particles with unit mass). A well known variational principle states that a curve x joining x_0 and x_1 is a brachistochrone of fixed energy E if and only if x is a geodesic with respect to the conformal Riemannian metric $\phi_E \cdot g_0$, with conformal factor $\phi_E = (E - V)^{-1}$.

The brachistochrone problem can also be formulated in the context of general relativity. We want to emphasize here that the original solution to the brachistochrone problem offered by Johann Bernoulli, which lacked mathematical rigor, can be made absolutely rigorous in a general relativistic context. Namely, the trajectory of a freely falling massive object, which is represented by a timelike geodesic in a Lorentzian manifold, is characterized by extremizing its arrival time measured by means of a smooth parameterization of the receiving observer. This is the so called general relativistic timelike Fermat Principle, suggested in [14] and rigorously proven in [6].

The first relativistic versions of the brachistochrone problem appear in [10] and [13]. V. Perlick (see [22]) has determined the brachistochrone equation in a stationary Lorentzian manifold of splitting type, and two of the authors, together with J. Verderesi, in [9] have generalized Perlick's result to the case of an arbitrary stationary manifold by reformulating the brachistochrone problem in the context of sub-Riemannian geometry. We recall that a stationary metric that satisfies the Einstein's equations describes a time-independent gravitational field in General Relativity.

The variational principle proven in [9] was then used in [7] to prove some results concerning the existence and the multiplicity of relativistic brachistochrones with a given value of energy between a fixed event and a fixed observer of a stationary spacetime.

We formulate the general relativistic brachistochrone problem for the travel time as follows.

Let (\mathcal{M}, g) be a 4-dimensional Lorentzian manifold, i.e., an arbitrary spacetime in the sense of general relativity and fix a timelike smooth vector field Y on \mathcal{M} . For simplicity, we assume that Y is complete, i.e., its integral lines are defined over the entire real line. The integral curves of Y can be interpreted as the worldlines of *observers*. Please note that we do not require Y to be normalized, i.e., in general the worldlines of our observers are not parameterized by proper time. The reason is that in the stationary case, i.e., if (\mathcal{M}, g) admits a timelike Killing vector field, it is convenient to choose this Killing vector field for Y and not a renormalized version of it.

To formulate the brachistochrone problem with respect to our arbitrarily chosen observer field Y , we fix a point p in \mathcal{M} , a (maximal) integral curve $\gamma : \mathbb{R} \mapsto \mathcal{M}$ of Y and a real number $k > 0$. The *trial paths* for our variational problem are all timelike smooth curves $\sigma : [0, 1] \mapsto \mathcal{M}$ which are nowhere tangent to Y and satisfy the following conditions:

$$(1.2) \quad \sigma(0) = p;$$

$$(1.3) \quad \sigma(1) \in \gamma(\mathbb{R});$$

$$(1.4) \quad g(\dot{\sigma}(0), Y(\sigma(0))) = -k \left(-g(\dot{\sigma}, \dot{\sigma}) \right)^{1/2};$$

$$(1.5) \quad g(\nabla_{\dot{\sigma}} \dot{\sigma}, \dot{\sigma}) = 0;$$

$$(1.6) \quad g(\nabla_{\dot{\sigma}} \dot{\sigma}, Y) = 0.$$

Here ∇ denotes the Levi-Civita connection of the Lorentzian metric g . We denote by $\mathcal{B}_{p,\gamma}(k)$ the set of trial paths; in the rest of the paper we will be working with suitable completions of this space.

If we interpret each integral curve of Y as a “point in space”, (1.2) and (1.3) mean that all trial paths connect the same two points in space, where the starting time is fixed whereas the

arrival time is not. Condition (1.4) says that all trial paths start with the same speed with respect to the observer field Y .

Observe that, in order to simplify the mathematics, we have chosen to parameterize our trial curves on the interval $[0, 1]$, rather than using a proper time parameterization over intervals varying with the curves. By condition (1.5), the quantity \mathcal{T}_σ defined by $-\mathcal{T}_\sigma^2 = g(\dot{\sigma}, \dot{\sigma})$ is a constant for each trial path σ (but takes different values for different trial paths). This implies that the curve parameter t along σ is related to proper time τ by an affine transformation, $\tau = \mathcal{T}_\sigma t + \text{const.}$ As a consequence, the 4-velocity along each trial path is given by $\mathcal{T}_\sigma^{-1} \dot{\sigma}$, whereas the 4-acceleration is given by $\mathcal{T}_\sigma^{-2} \nabla_{\dot{\sigma}} \dot{\sigma}$. Hence, conditions (1.5) and (1.6) require the 4-acceleration to be perpendicular to the plane spanned by $\dot{\sigma}$ and Y . In other words, with respect to the observer field Y there are only forces perpendicular to the direction of motion. Such forces can be interpreted as constraint forces supplied by a frictionless slide which is at rest with respect to the observer field Y .

The brachistochrone problem can now be formulated in the following way.

Among all trial paths that satisfy the above-mentioned conditions, we want to find those curves for which the travel time is minimal or, more generally, stationary.

A different general relativistic brachistochrone problem can be formulated, by requiring that the solutions be stationary points for the *arrival time* functional, given by $AT(\sigma) = \gamma^{-1}(\sigma(1))$. In other words, $AT(\sigma)$ is the value of the proper time of the receiver at the arrival event. In physical terms, the two brachistochrone problems differ by the way of measuring time: in the first case the time is measured by a watch traveling along the trajectory of the mass, in the second case the time is measured by the observer that receives the mass at the end of its trajectory. The two variational problems are essentially different; in this paper we stick to the first problem, while the "arrival time brachistochrones" are the subject of a followup paper.

If (\mathcal{M}, g) is a stationary spacetime and Y is a *Killing* vector field, i.e., the flow of Y preserves the metric g , then the condition (1.6) means that the product $g(\dot{\sigma}, Y)$ is constant along σ . The value of this constant can be easily computed using condition (1.4), that gives $g(\dot{\sigma}, Y) \equiv -k\mathcal{T}_\sigma$. Hence, in the stationary case, the conditions (1.4) and (1.6) can be resumed in the condition:

$$(1.7) \quad g(\dot{\sigma}, Y) = -k\mathcal{T}_\sigma.$$

Again, observe that the value of the travel time \mathcal{T}_σ appears in formula (1.7) because of our choice of the parameterization on the interval $[0, 1]$ of our trial curves. The condition (1.7) is the relativistic counterpart of the energy conservation law (1.1) in the Newtonian case. Although physically meaningful, the mathematical approach to the general relativistic brachistochrone problem in the non stationary case presents difficulties of higher order than in the stationary case. For instance, it is not even clear whether the non stationary brachistochrones are solutions to a second order differential equation; in Reference [23], the authors used a Lagrange multiplier technique to derive a system of differential equations for the brachistochrones and for the Lagrangian multipliers. Unfortunately, it does not seem to be possible to eliminate the Lagrangian multipliers from the system without introducing integrals, unless in the stationary case. Thus, it looks as if the brachistochrones in the non-stationary case are not determined by a second-order differential equation, but rather by an integro-differential equation.

For these technical reasons, in this paper we will only study the case of a manifold \mathcal{M} with metric g which is stationary with respect to the observer field Y .

The purpose of this article is to present a complete variational theory for travel time brachistochrones in a stationary Lorentzian manifold and, in particular, it will be developed a full-fledged infinite dimensional Morse theory for the critical points of the travel time.

We present below a list of the main results proven in this paper:

- the general-relativistic brachistochrone problem in a stationary Lorentzian manifold is presented in a context of Global Analysis on infinite dimensional Hilbertian manifolds (Section 2);

- the travel time brachistochrones are smooth curves; they can be characterized as the only solutions of a second order differential equation (formula (3.21) and Proposition 4.1);
- the brachistochrones can also be characterized as local minimizers for the travel time, and, equivalently, as curves whose *spatial* part is a geodesic with respect to a suitable Riemannian structure on \mathcal{M} (Proposition 4.5);
- it is computed a second order variation formula for the travel time functional, which is characterized by a Morse Index Theorem (Theorem 7.12). In analogy with the Riemannian geodesic problem, this theorem relates the nature of a stationary point for the travel time with some metrical properties of \mathcal{M} and with the *convexity* of the timelike curve γ representing the observer and measured by the second fundamental form of γ ;
- under suitable completeness hypotheses for \mathcal{M} , we prove the global Morse relations for the travel time functional in a completion of the space $\mathcal{B}_{p,\gamma}(k)$ (Section 8); thanks to this relations one obtains estimates on the number of brachistochrones of fixed energy k between p and γ , according to the topology and the metric of \mathcal{M} .

From a strictly mathematical point of view, the paper presents some technicalities that is worth discussing. The main difficulties in our variational problem are due to the presence of the double constraint given by (1.5) and (1.6) (or (1.7)), which are, respectively, quadratic and linear in the first derivative.

Due to this kind of constraint, in order to put a differentiable structure on the set $\mathcal{B}_{p,\gamma}(k)$ of trial paths, one needs to consider a Hilbert space completion of $\mathcal{B}_{p,\gamma}(k)$ made in a Sobolev space of curves having at least the C^1 -regularity, and thus one is forced to consider curves of class H^2 (see formula (2.16), Proposition 2.1 and Remark 2.7).

However, the H^2 -approach has the disadvantage of introducing new difficulties, especially for the following reasons:

- the Riesz duality in the Hilbert spaces H^i involves products of functions and also their derivatives, resulting in lengthy and complicated calculations when using the Lagrange multipliers method;
- the arrival time functional *does not* satisfy good compactness properties in the space of H^2 -curves, like the Palais–Smale condition (see Appendix B), which is an essential tool for developing an infinite dimensional Morse Theory.

The problem of duality in Hilbert spaces of curves with *high* regularity is faced through the introduction of a suitable formalism based on the theory of *distribution* and *generalized functions*, whose technical details are worked out at the beginning of Section 3. Unavoidably, the results needed are stated and proven in a formal way, and this part of the paper turns out to be rather technical nature. Even though these results are essential from a formal point of view, the reader should not be intimidated by Proposition 3.1 and the few subsequent Lemmas, and should keep his/her attention to the main issue of the paper.

As to the problem of lack of compactness for the travel time functional, the crucial observation here is that, if one is only interested in a local differentiable structure, then around each smooth (C^2) curve σ in $\mathcal{B}_{p,\gamma}(k)$ it can be defined a differentiable chart on the set of H^1 -curves that are uniformly close to σ (see Proposition 2.8). Since the solution to our variational problem are proven to be curves of class C^2 , then one can relax the requirement of convergence for the Palais–Smale sequences, which allows to prove the global Morse relations for the arrival time functional (Section 8).

The paper is organized according to the following outline.

In Section 2 we discuss the variational setup, where we define our main function spaces and functionals, proving their differentiability in the setting of infinite dimensional Hilbertian manifolds.

In Section 3 we present a Lagrange multiplier approach to the brachistochrone problem, and we derive some conditions on the curves that are extrema for the travel time functional and

their corresponding multipliers. Moreover, we obtain a differential equation that is satisfied by the brachistochrones.

Section 4 is devoted to the proof of the variational principle for brachistochrones, that extends the principle proven in [9] for local minimizers of the travel time. We also prove that the differential equation determined in Section 3 is the equation obtained by the above variational principle.

In Section 5 we study the second variation of the travel time T at a given brachistochrone. We prove a second order variational principle for brachistochrones, that relates the Hessian H^T to the Hessian of the energy functional of a suitable Riemannian metric on \mathcal{M} .

In Section 6 we recall some known facts about the Morse Index Theorem for orthogonal geodesics between submanifolds in Riemannian geometry, and we prove a slightly different version of the theorem for the case of a manifold admitting a Killing vector field. This result (Theorem 6.9), which has some interest on its own and for this reason it is stated in a general form, is then used in the next section to prove a brachistochrone version of the Morse Index Theorem.

In Section 7, in analogy with the classical Morse theory for Riemannian geodesics, we define the notions of Jacobi fields and focal points along a brachistochrone, and we prove a version of the Morse Index Theorem for brachistochrones. Some immediate consequence of the theory concerning the local nature of the critical points of T are derived.

Section 8 is dedicated to the proof of the global Morse relations, from which we obtain some results on the multiplicity of brachistochrones with a given value of the energy between an event and an observer.

Finally, the paper has two short appendices containing some side results. In Appendix A we show the explicit calculation of the second variation of the travel time functional at a given brachistochrone. In Appendix B we discuss a simple but instructive example to show that the travel time functional does not satisfy the Palais–Smale condition in the space of curves satisfying an H^2 -regularity condition.

2. THE FUNCTIONAL SPACES AND THE VARIATIONAL SETUP

Throughout this paper we will denote by (\mathcal{M}, g) a stationary Lorentzian manifold, with g a Lorentzian metric tensor on \mathcal{M} , and Y is a smooth timelike Killing vector field on \mathcal{M} , which is assumed to be complete.

The symbol $\langle \cdot, \cdot \rangle$ will denote the bilinear form induced by g on the tangent spaces of \mathcal{M} ; the usual nabla symbol ∇ will denote the covariant derivative relative to the Levi–Civita connection of g . Given a smooth function ϕ on \mathcal{M} , for $q \in \mathcal{M}$ we denote by $\nabla\phi(q)$ the gradient of ϕ at q with respect to g , which is the vector in $T_q\mathcal{M}$ defined by $\langle \nabla\phi(q), \cdot \rangle = d\phi(q)[\cdot]$; the Hessian $H^\phi(q)$ of ϕ at q is the symmetric bilinear form on $T_q\mathcal{M}$ given by $H^\phi(q)[v_1, v_2] = \langle \nabla_{v_1}\nabla\phi, v_2 \rangle$, for $v_1, v_2 \in T_q\mathcal{M}$.

We denote by $\psi : \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M}$ the flow of Y , i.e., for $q \in \mathcal{M}$ and $t \in \mathbb{R}$, $\psi(q, t)$ is the value $\gamma_q(t)$, where γ_q is the maximal integral line of Y satisfying $\gamma_q(0) = q$. The Killing property of Y , which is crucial in most of the results presented in this paper, will be used systematically in our computations through the following three facts:

1. the quantity $\langle Y, Y \rangle$ is constant along the flow lines of Y ,
2. the differential $d_x\psi(q, t_0) : T_q\mathcal{M} \rightarrow T_{\psi(q, t_0)}\mathcal{M}$ of the map $\psi(\cdot, t_0)$ is an isometry for all t_0 , or, equivalently, for all t_0 the map $q \mapsto \psi(q, t)$ is a local isometry of \mathcal{M} ;
3. $\langle \nabla_v Y, w \rangle = -\langle \nabla_w Y, v \rangle$ for all pair of vectors v and w ; in particular, for all $v \in T\mathcal{M}$, we have $\langle \nabla_v Y, v \rangle = 0$.

Observe that the second or the third condition above is in fact *equivalent* to the Killing property of Y (see [19, Proposition 9.25]).

We set:

$$m = \dim(\mathcal{M});$$

the physical interesting case is $m = 4$.

We denote by R the *curvature tensor* of the Lorentzian metric g , with the following sign convention:

$$(2.1) \quad R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}.$$

for X, Y vector fields on \mathcal{M} .

As customary, for $1 \leq p \leq +\infty$, $L^p([0, 1], \mathbb{R})$ will denote the space of Lebesgue p -integrable real functions; for $n \in \mathbb{N}$, $H^n([0, 1], \mathbb{R})$ will denote the Sobolev space of functions of class C^{n-1} and having weak n -th derivative in $L^2([0, 1], \mathbb{R})$.

We introduce for convenience the auxiliary Riemannian metric g_R on \mathcal{M} , given by:

$$(2.2) \quad g_R(p)[v_1, v_2] = \langle v_1, v_2 \rangle_{(R)} = \langle v_1, v_2 \rangle - 2 \frac{\langle v_1, Y(q) \rangle \cdot \langle v_2, Y(q) \rangle}{\langle Y(q), Y(q) \rangle},$$

for $q \in \mathcal{M}$ and $v_1, v_2 \in T_q \mathcal{M}$. It is easy to see that Y is Killing also in the metric g_R ; moreover, the restriction of g and g_R on the orthocomplement of Y coincide.

We define the space $L^2([0, 1], T\mathcal{M})$ of square integrable $T\mathcal{M}$ -valued functions:

$$(2.3) \quad L^2([0, 1], T\mathcal{M}) = \left\{ \zeta : [0, 1] \mapsto T\mathcal{M} \text{ measurable} : \int_0^1 \langle \zeta(t), \zeta(t) \rangle_{(R)} dt < +\infty \right\}.$$

Let $\pi : T\mathcal{M} \mapsto \mathcal{M}$ be the canonical projection. Given any curve $\sigma : I \subseteq \mathbb{R} \mapsto A$, a *vector field along* σ is a map $\zeta : I \mapsto T\mathcal{M}$ such that $\pi \circ \zeta = \sigma$. Let A be any open set of \mathcal{M} ; the Sobolev space $H^1([0, 1], A)$ is defined by:

$$(2.4) \quad H^1([0, 1], A) = \left\{ \sigma : [0, 1] \mapsto A : \sigma \text{ absolutely continuous, } \dot{\sigma} \in L^2([0, 1], T\mathcal{M}) \right\}.$$

For $A \subseteq \mathcal{M}$, the symbol $C^1([0, 1], A)$ will denote the set of C^1 -curves defined $[0, 1]$ and with image in A ; we also define the Sobolev space $H^2([0, 1], A)$ as:

$$(2.5) \quad H^2([0, 1], A) = \left\{ \sigma \in C^1([0, 1], A) : \dot{\sigma} \text{ is absolutely continuous, and } \nabla_{\dot{\sigma}} \dot{\sigma} \in L^2([0, 1], T\mathcal{M}) \right\}.$$

It is not too difficult to prove that the definition of the spaces $H^i([0, 1], A)$ does *not* indeed depend on the choice of the Riemannian metric g_R , nor on the choice of the linear connection ∇ that appears in (2.5). As a matter of fact, the spaces $H^i([0, 1], A)$ can be defined intrinsically for any differentiable manifold A using local charts (see [20]) or, equivalently, using auxiliary structures on A , like for instance a Riemannian metric. In the sequel, we will use the spaces $H^i([0, 1], A)$, $i = 1, 2$, where A will be an open subset of \mathcal{M} or $T\mathcal{M}$.

If A is a smooth submanifold of \mathcal{M} , in particular if A is an open subset, then $H^i([0, 1], A)$ has the structure of an infinite dimensional Hilbertian manifold, modeled on the Sobolev space $H^i([0, 1], \mathbb{R}^m)$; for $\sigma \in H^i([0, 1], A)$, the tangent space $T_\sigma H^i([0, 1], A)$ can be identified with the Hilbert space:

$$(2.6) \quad T_\sigma H^i([0, 1], A) = \left\{ \zeta \in H^i([0, 1], T\mathcal{M}) : \zeta \text{ vector field along } \sigma \right\}.$$

The inner product in $T_\sigma H^1([0, 1], A)$ is given by:

$$(2.7) \quad \langle \zeta, \zeta \rangle_* = \int_0^1 \left(\langle \zeta, \zeta \rangle_{(R)} + \langle \nabla_{\dot{\sigma}} \zeta, \nabla_{\dot{\sigma}} \zeta \rangle_{(R)} \right) dt,$$

while the inner product in $T_\sigma H^2([0, 1], A)$ is given by:

$$(2.8) \quad \langle \zeta, \zeta \rangle_{**} = \int_0^1 \left(\langle \zeta, \zeta \rangle_{(R)} + \langle \nabla_{\dot{\sigma}} \zeta, \nabla_{\dot{\sigma}} \zeta \rangle_{(R)} + \langle \nabla_{\dot{\sigma}}^2 \zeta, \nabla_{\dot{\sigma}}^2 \zeta \rangle_{(R)} \right) dt,$$

where $\nabla_{\dot{\sigma}}^2 \zeta = \nabla_{\dot{\sigma}}(\nabla_{\dot{\sigma}} \zeta)$.

Let k be a fixed positive constant, with $-k^2 < \sup_{\mathcal{M}} \langle Y(q), Y(q) \rangle$, and U_k be the open set:

$$(2.9) \quad U_k = \left\{ q \in \mathcal{M} : \langle Y(q), Y(q) \rangle + k^2 > 0 \right\}.$$

Since Y is Killing, the quantity $\langle Y, Y \rangle$ is constant along the integral lines of Y , hence U_k is invariant by the flow of Y .

We will denote by p a fixed event of U_k and by $\gamma : \mathbb{R} \mapsto U_k$ a given integral line of Y which does not pass through p . We introduce the spaces

$$(2.10) \quad \Omega_{p,\gamma}^{(i)} = \Omega_{p,\gamma}^{(i)}(U_k) = \left\{ w \in H^i([0, 1], U_k) : w(0) = p, w(1) \in \gamma(\mathbb{R}) \right\}, \quad i = 1, 2.$$

It is well known that $\Omega_{p,\gamma}^{(i)}$ is a smooth submanifold of $H^i([0, 1], U_k)$; for $w \in \Omega_{p,\gamma}^{(i)}$, the tangent space $T_w \Omega_{p,\gamma}^{(i)}$ is given by:

$$(2.11) \quad T_w \Omega_{p,\gamma}^{(i)} = \left\{ \zeta \in T_w H^i([0, 1], U_k) : \zeta(0) = 0, \zeta(1) \in \mathbb{R} \cdot Y(w(1)) \right\}.$$

For $w \in \Omega_{p,\gamma}^{(i)}$, $T_w \Omega_{p,\gamma}^{(i)}$ is a Hilbert space with respect to the inner products:

$$(2.12) \quad \langle \zeta, \zeta \rangle_1 = \int_0^1 \langle \nabla_{\dot{w}} \zeta, \nabla_{\dot{w}} \zeta \rangle_{(R)} dt$$

in the case of $T_w \Omega_{p,\gamma}^{(1)}$ and

$$(2.13) \quad \langle \zeta, \zeta \rangle_2 = \int_0^1 \left(\langle \nabla_{\dot{w}} \zeta, \nabla_{\dot{w}} \zeta \rangle_{(R)} + \langle \nabla_{\dot{w}}^2 \zeta, \nabla_{\dot{w}}^2 \zeta \rangle_{(R)} \right) dt$$

for $T_w \Omega_{p,\gamma}^{(2)}$. Observe that, since $\zeta(0) = 0$ for all $\zeta \in T_w \Omega_{p,\gamma}^{(i)}$, then the inner products $\langle \cdot, \cdot \rangle_*$ and $\langle \cdot, \cdot \rangle_{**}$ of formulas (2.7) and (2.8) are equivalent in $T_w \Omega_{p,\gamma}^{(2)}$, respectively, to the products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ of formulas (2.12) and (2.13).

We consider the *action* functional F on $\Omega_{p,\gamma}^{(i)}$, given by:

$$(2.14) \quad F(\sigma) = \frac{1}{2} \int_0^1 \langle \dot{\sigma}, \dot{\sigma} \rangle dt.$$

It is well known that F is smooth; for $\sigma \in \Omega_{p,\gamma}^{(i)}$ and $V \in T_\sigma \Omega_{p,\gamma}^{(i)}$, the Gateaux derivative $dF(\sigma)[V]$ is given by:

$$(2.15) \quad dF(\sigma)[V] = \int_0^1 \langle \nabla_{\dot{\sigma}} V, \dot{\sigma} \rangle dt.$$

Finally, for all positive constant $k \in \mathbb{R}^+$, we introduce the spaces $\mathcal{B}_{p,\gamma}^{(i)}(k)$, $i = 1, 2$, by:

$$(2.16) \quad \mathcal{B}_{p,\gamma}^{(i)}(k) = \left\{ \sigma \in \Omega_{p,\gamma}^{(i)} : \exists \mathcal{T}_\sigma \in \mathbb{R}^+ \text{ such that } \langle \dot{\sigma}, Y \rangle \equiv -k \mathcal{T}_\sigma \text{ and } \langle \dot{\sigma}, \dot{\sigma} \rangle \equiv -\mathcal{T}_\sigma^2 \right\}.$$

We define the *travel time functional* T on $\mathcal{B}_{p,\gamma}^{(i)}(k)$ by:

$$(2.17) \quad T(\sigma) = \mathcal{T}_\sigma.$$

The main goal of this section is to establish an infinite dimensional differentiable structure on the sets $\mathcal{B}_{p,\gamma}^{(2)}(k)$ and $\mathcal{B}_{p,\gamma}^{(1)}(k)$. The case of $\mathcal{B}_{p,\gamma}^{(2)}(k)$ is easier, and its regularity is proven in the next Proposition. For the set $\mathcal{B}_{p,\gamma}^{(1)}(k)$, we are only able to establish its regularity around some special points; this second case is treated at the end of this section.

Proposition 2.1. $\mathcal{B}_{p,\gamma}^{(2)}(k)$ is a smooth submanifold of $\Omega_{p,\gamma}^{(2)}$. For $\sigma \in \mathcal{B}_{p,\gamma}^{(2)}(k)$, the tangent space $T_\sigma \mathcal{B}_{p,\gamma}^{(2)}(k)$ can be identified with the Hilbert space:

$$(2.18) \quad T_\sigma \mathcal{B}_{p,\gamma}^{(2)}(k) = \left\{ \zeta \in T_\sigma \Omega_{p,\gamma}^{(2)} : \exists C_\zeta \in \mathbb{R} \text{ such that } \langle \nabla_{\dot{\sigma}} \zeta, Y \rangle - \langle \zeta, \nabla_{\dot{\sigma}} Y \rangle \equiv C_\zeta \text{ and } \langle \nabla_{\dot{\sigma}} \zeta, \dot{\sigma} \rangle \equiv \frac{\mathcal{T}_\sigma C_\zeta}{k} \right\},$$

endowed with the inner product $\langle \cdot, \cdot \rangle_2$ of formula (2.13).

Proof. For $\sigma \in \Omega_{p,\gamma}^{(2)}$, the maps $\langle \dot{\sigma}, Y \rangle$, $\langle \dot{\sigma}, Y \rangle^2$ and $\langle \dot{\sigma}, \dot{\sigma} \rangle$ are in $H^1([0, 1], \mathbb{R})$. Let $k \in \mathbb{R}^+$ be a fixed constant. We consider the following map:

$$(2.19) \quad \mathcal{F} : \Omega_{p,\gamma}^{(2)} \longmapsto H^1([0, 1], \mathbb{R}) \times H^1([0, 1], \mathbb{R})$$

given by:

$$(2.20) \quad \mathcal{F}(\sigma) = (\langle \dot{\sigma}, Y \rangle, \langle \dot{\sigma}, Y \rangle^2 + k^2 \langle \dot{\sigma}, \dot{\sigma} \rangle).$$

It is not difficult to prove that \mathcal{F} is a smooth map and that, for $\sigma \in \Omega_{p,\gamma}^{(2)}$ and $V \in T_\sigma \Omega_{p,\gamma}^{(2)}$, the Gateaux derivative $d\mathcal{F}(\sigma)[V]$ is given by:

$$(2.21) \quad d\mathcal{F}(\sigma)[V] = (\langle \nabla_{\dot{\sigma}} V, Y \rangle - \langle V, \nabla_{\dot{\sigma}} Y \rangle, 2\langle \dot{\sigma}, Y \rangle(\langle \nabla_{\dot{\sigma}} V, Y \rangle - \langle V, \nabla_{\dot{\sigma}} Y \rangle) + 2k^2 \langle \nabla_{\dot{\sigma}} V, \dot{\sigma} \rangle).$$

Here we have used the fact that Y is Killing, thus $\langle \dot{\sigma}, \nabla_V Y \rangle = -\langle V, \nabla_{\dot{\sigma}} Y \rangle$.

Let \mathcal{C} denote the subspace of $H^1([0, 1], \mathbb{R})$ given by the constant functions, and let \mathcal{C}^- denote the open submanifold of \mathcal{C} consisting of negative functions:

$$(2.22) \quad \begin{aligned} \mathcal{C} &= \left\{ h \in H^1([0, 1], \mathbb{R}) : h \equiv h_0 \text{ (const.) a. e.} \right\}, \\ \mathcal{C}^- &= \left\{ h \in \mathcal{C} : h < 0 \text{ a. e.} \right\}. \end{aligned}$$

It is easy to see that $\mathcal{B}_{p,\gamma}^{(2)}(k) = \mathcal{F}^{-1}(\mathcal{C}^- \times \{0\})$.

Let $\tilde{H}^1([0, 1], \mathbb{R})$ denote the quotient space $H^1([0, 1], \mathbb{R})/\mathcal{C}$, which is naturally identified with the set of functions with null average in $[0, 1]$.

Let Π be the map:

$$(2.23) \quad \Pi : H^1([0, 1], \mathbb{R}) \times H^1([0, 1], \mathbb{R}) \longmapsto \tilde{H}^1([0, 1], \mathbb{R}) \times H^1([0, 1], \mathbb{R})$$

given by the quotient map on the first factor and the identity on the second factor.

To prove the Proposition we use the *Inverse Mapping Theorem* (see [15]). According to this Theorem, $\mathcal{B}_{p,\gamma}^{(2)}(k)$ is a smooth submanifold of $\Omega_{p,\gamma}^{(2)}$ provided that the map \mathcal{F} be *transversal* over $\mathcal{C}^- \times \{0\}$, i.e., if for all $\sigma \in \mathcal{B}_{p,\gamma}^{(2)}(k)$ the composite map:

$$(2.24) \quad \Pi \circ d\mathcal{F}(\sigma) : T_\sigma \Omega_{p,\gamma}^{(2)} \longmapsto \tilde{H}^1([0, 1], \mathbb{R}) \times H^1([0, 1], \mathbb{R})$$

is surjective. This amounts to saying that, for all $h_1, h_2 \in H^1([0, 1], \mathbb{R})$ there exists a constant $c \in \mathbb{R}$ such that the system of differential equations:

$$(2.25) \quad \langle \nabla_{\dot{\sigma}} V, Y \rangle - \langle V, \nabla_{\dot{\sigma}} Y \rangle = h_1 + c$$

$$(2.26) \quad 2\langle \dot{\sigma}, Y \rangle(\langle \nabla_{\dot{\sigma}} V, Y \rangle - \langle V, \nabla_{\dot{\sigma}} Y \rangle) + 2k^2 \langle \nabla_{\dot{\sigma}} V, \dot{\sigma} \rangle = h_2$$

has at least one solution $V \in T_\sigma \Omega_{p,\gamma}^{(2)}$. Using the fact that $\langle \dot{\sigma}, Y \rangle \equiv -k\mathcal{T}_\sigma$, we can rewrite (2.26) as:

$$(2.27) \quad \langle \nabla_{\dot{\sigma}} V, \dot{\sigma} \rangle = h_3,$$

where

$$h_3 = \frac{h_2 + 2k\mathcal{T}_\sigma(h_1 + c)}{2k^2}$$

is in $H^1([0, 1], \mathbb{R})$.

Let $Z \in H^2([0, 1], T\mathcal{M})$ be a vector field along σ satisfying

$$(2.28) \quad \langle Y, Z \rangle \equiv 0, \quad \text{and} \quad \langle Z, \dot{\sigma} \rangle \neq 0.$$

To prove the existence of such a vector field Z , consider first the vector field along σ given by $\dot{\sigma}^\perp$, which is the orthogonal projection of $\dot{\sigma}$ onto the distribution $\Delta = Y^\perp$ orthogonal to Y . Formally, we have:

$$(2.29) \quad \dot{\sigma}^\perp = \dot{\sigma} - \frac{\langle \dot{\sigma}, Y \rangle}{\langle Y, Y \rangle} Y = \dot{\sigma} + \frac{k \mathcal{T}_\sigma}{\langle Y, Y \rangle} Y.$$

Obviously, we have:

$$(2.30) \quad \langle \dot{\sigma}^\perp, \dot{\sigma} \rangle = -\mathcal{T}_\sigma^2 \frac{k^2 + \langle Y, Y \rangle}{\langle Y, Y \rangle} \neq 0,$$

because $k^2 + \langle Y, Y \rangle \neq 0$ in U_k .

Observe that $\dot{\sigma}^\perp \in H^1$, and it does not have the required H^2 -regularity. Now, let Z be any section of class H^2 of Δ which is *uniformly* close to $\dot{\sigma}^\perp$, in such a way that $\langle Z, \dot{\sigma} \rangle \neq 0$ as well.¹ Observe in particular that, since $\langle Z, \dot{\sigma} \rangle$ is continuous, then $\langle Z, \dot{\sigma} \rangle^{-1}$ is in $L^\infty([0, 1], \mathbb{R})$.

In order to solve equations (2.25) and (2.27), we set

$$V = aY + bZ,$$

where $a, b \in H^2([0, 1], \mathbb{R})$ are to be determined. Observe that such a V belongs to $T_\sigma \Omega_{p, \gamma}^{(2)}$ provided that a and b satisfy the boundary conditions:

$$(2.31) \quad a(0) = b(0) = 0, \quad \text{and} \quad b(1) = 0.$$

Since $\langle Z, Y \rangle = 0$, equations (2.25) and (2.27) are translated into:

$$(2.32) \quad a' \langle Y, Y \rangle + 2b \langle \nabla_{\dot{\sigma}} Z, Y \rangle = h_1 + c$$

$$(2.33) \quad -a' k \mathcal{T}_\sigma + b' \langle Z, \dot{\sigma} \rangle + b \langle \nabla_{\dot{\sigma}} Z, \dot{\sigma} \rangle = h_3.$$

We solve for a' equation (2.32) obtaining:

$$(2.34) \quad a' = \langle Y, Y \rangle^{-1} [h_1 + c - 2b \langle \nabla_{\dot{\sigma}} Z, Y \rangle];$$

substituting (2.34) in (2.33) gives:

$$(2.35) \quad b' + \alpha b = \beta + c\gamma,$$

where

$$\alpha = \frac{\langle Y, Y \rangle \langle \nabla_{\dot{\sigma}} Z, \dot{\sigma} \rangle + 2k \mathcal{T}_\sigma \langle \nabla_{\dot{\sigma}} Z, Y \rangle}{\langle Z, \dot{\sigma} \rangle \langle Y, Y \rangle},$$

and

$$\beta = \frac{k \mathcal{T}_\sigma h_1 + h_3 \langle Y, Y \rangle}{\langle Z, \dot{\sigma} \rangle \langle Y, Y \rangle}, \quad \gamma = \frac{k \mathcal{T}_\sigma}{\langle Z, \dot{\sigma} \rangle \langle Y, Y \rangle}.$$

Observe that α, β and γ are in $H^1([0, 1], \mathbb{R})$. Thus, the unique solution b of (2.35) satisfying $b(0) = 0$, given by:

$$(2.36) \quad b(t) = e^{-\int_0^t \alpha} \left[\int_0^t \beta e^{\int_0^s \alpha} + c \int_0^t \gamma e^{\int_0^s \alpha} \right],$$

is in $H^2([0, 1], \mathbb{R})$. Observe that $\gamma \neq 0$ in $[0, 1]$, and so $\int_0^1 \gamma e^{\int_0^s \alpha} \neq 0$. In particular, there exists $c \in \mathbb{R}$ such that $b(1) = 0$.

Finally, a can be chosen as the unique solution of (2.34) satisfying $a(0) = 0$. Observe that the right hand side of (2.34) is in $H^1([0, 1], \mathbb{R})$, so $a \in H^2([0, 1], \mathbb{R})$ and \mathcal{F} is transversal over \mathcal{C}^- . Hence, $\mathcal{B}_{p, \gamma}^{(2)}(k)$ is a smooth submanifold of $\Omega_{p, \gamma}^{(2)}$.

¹For the approximation theorem, we can use an H^2 parallel referential of Δ along σ , so that sections of Δ along σ will be identified with curves in the Euclidean space, and standard approximation results apply.

By the Inverse Mapping Theorem, for $\sigma \in \mathcal{B}_{p,\gamma}^{(2)}(k)$, the tangent space $T_\sigma \mathcal{B}_{p,\gamma}^{(2)}(k)$ is identified with the kernel of the map $\Pi \circ d\mathcal{F}(\sigma)$, which consists of the vector fields $\zeta \in T_\sigma \Omega_{p,\gamma}^{(2)}$ such that $d\mathcal{F}(\sigma)[\zeta] \in \mathcal{C} \times \{0\}$.

Recalling (2.25) and (2.26), we have that $\zeta \in T_\sigma \Omega_{p,\gamma}^{(2)}$ belongs to $T_\sigma \mathcal{B}_{p,\gamma}^{(2)}(k)$ if and only if there exists $C_\zeta \in \mathbb{R}$ such that ζ satisfies the equations:

$$(2.37) \quad \langle \nabla_{\dot{\sigma}} \zeta, Y \rangle - \langle \zeta, \nabla_{\dot{\sigma}} Y \rangle = C_\zeta,$$

$$(2.38) \quad -2k \mathcal{T}_\sigma C_\zeta + 2k^2 \langle \nabla_{\dot{\sigma}} \zeta, \dot{\sigma} \rangle = 0.$$

From (2.37) and (2.38) we easily obtain (2.18) and we are done. \square

Given a curve $\sigma \in \mathcal{B}_{p,\gamma}^{(2)}(k)$, a vector field $\zeta \in T_\sigma \mathcal{B}_{p,\gamma}^{(2)}(k)$ will be called a *variational vector field* along σ .

In some parts of the paper (see Section 7) we will need to consider variations of curves in $\mathcal{B}_{p,\gamma}^{(2)}(k)$ by curves σ satisfying the conditions (1.6) and (1.7), but not necessarily with endpoints in p and γ . For this reason, for $i = 1, 2$ we introduce the sets:

$$(2.39) \quad \mathcal{B}_p^{(i)}(k) = \bigcup_{\gamma \subset U_k} \mathcal{B}_{p,\gamma}^{(i)}(k), \quad \text{and} \quad \mathcal{B}^{(i)}(k) = \bigcup_{p,\gamma \subset U_k} \mathcal{B}_{p,\gamma}^{(i)}(k),$$

where the unions in (2.39) are taken over all γ 's that are integral lines of Y having image in U_k .

Using the same argument of Proposition 2.1, it is an easy exercise to prove that both $\mathcal{B}_p^{(2)}(k)$ and $\mathcal{B}^{(2)}(k)$ are smooth Hilbert submanifolds of $H^2([0, 1], U_k)$ and that, for $\sigma \in \mathcal{B}_p^{(2)}(k)$ or $\sigma \in \mathcal{B}^{(2)}(k)$, the tangent spaces $T_\sigma \mathcal{B}_p^{(2)}(k)$ and $T_\sigma \mathcal{B}^{(2)}(k)$ are Hilbert subspaces of $T_\sigma H^2([0, 1], U_k)$ given by:

$$(2.40) \quad T_\sigma \mathcal{B}_p^{(2)}(k) = \left\{ \zeta \in T_\sigma H^2([0, 1], U_k) : \zeta(0) = 0, \exists C_\zeta \in \mathbb{R} \text{ such that} \right. \\ \left. \langle \nabla_{\dot{\sigma}} \zeta, Y \rangle - \langle \zeta, \nabla_{\dot{\sigma}} Y \rangle \equiv C_\zeta \text{ and } \langle \nabla_{\dot{\sigma}} \zeta, \dot{\sigma} \rangle \equiv \frac{\mathcal{T}_\sigma C_\zeta}{k} \right\}.$$

and

$$(2.41) \quad T_\sigma \mathcal{B}^{(2)}(k) = \left\{ \zeta \in T_\sigma H^2([0, 1], U_k) : \exists C_\zeta \in \mathbb{R} \text{ such that} \right. \\ \left. \langle \nabla_{\dot{\sigma}} \zeta, Y \rangle - \langle \zeta, \nabla_{\dot{\sigma}} Y \rangle \equiv C_\zeta \text{ and } \langle \nabla_{\dot{\sigma}} \zeta, \dot{\sigma} \rangle \equiv \frac{\mathcal{T}_\sigma C_\zeta}{k} \right\}.$$

We restrict the action functional F of (2.14) to $\mathcal{B}_{p,\gamma}^{(2)}(k)$, obtaining the following:

Corollary 2.2. *The Gateaux derivative $dT(\sigma)[\zeta]$ of the travel time functional on $\mathcal{B}_{p,\gamma}^{(2)}(k)$ is given by:*

$$(2.42) \quad dT(\sigma)[\zeta] = -\frac{C_\zeta}{k}.$$

Proof. Since $\mathcal{B}_{p,\gamma}^{(2)}(k)$ is a smooth submanifold of $\Omega_{p,\gamma}^{(2)}$, then the restriction of the action functional F to $\mathcal{B}_{p,\gamma}^{(2)}(k)$ is smooth. For $\sigma \in \mathcal{B}_{p,\gamma}^{(2)}(k)$, we have:

$$(2.43) \quad F(\sigma) = -\frac{1}{2} \mathcal{T}_\sigma^2 < 0,$$

hence $T(\sigma) = \sqrt{-2F(\sigma)}$ is also smooth.

Equality (2.42) follows easily by differentiating the expression $\mathcal{T}_\sigma = -k^{-1} \langle \dot{\sigma}, Y \rangle$ and using the equality $C_\zeta = \langle \nabla_{\dot{\sigma}} \zeta, Y \rangle - \langle \zeta, \nabla_{\dot{\sigma}} Y \rangle$. \square

After setting up our variational framework, we are ready to give the following definition:

Definition 2.3. A *brachistochrone* of energy k between p and γ is a stationary point for the travel time functional T on $\mathcal{B}_{p,\gamma}^{(2)}(k)$. A brachistochrone curve σ is said to be *minimal* if σ is a minimum point for T on $\mathcal{B}_{p,\gamma}^{(2)}(k)$.

From Corollary 2.2 and Definition 2.3 we obtain immediately:

Corollary 2.4. A curve $\sigma \in \mathcal{B}_{p,\gamma}^{(2)}(k)$ is a brachistochrone of energy k between p and γ if and only if for every $\zeta \in T_\sigma \mathcal{B}_{p,\gamma}^{(2)}(k)$ it is $C_\zeta = \langle \nabla_{\dot{\sigma}} \zeta, Y \rangle - \langle \zeta, \nabla_{\dot{\sigma}} Y \rangle = 0$. \square

Remark 2.5. Observe that the definition of brachistochrone of energy k given in Definition 2.3 is different from the one given in ([9], Definition 1.1) and used in Reference [7]. Namely, in these articles, it was not established a differentiable structure in the set of admissible curves for the variational problem, and the brachistochrones of energy k were defined as curves locally minimizing their travel time. The equivalence of the two definitions will be given in Section 4, where we prove that the two approaches lead to exactly the same solutions.

Remark 2.6. Since T is strictly positive on $\mathcal{B}_{p,\gamma}^{(2)}(k)$, then its critical points coincide with the critical points in $\mathcal{B}_{p,\gamma}^{(2)}(k)$ of the restriction of the action functional $F = -\frac{1}{2}T^2$. The minimal brachistochrones of energy k are *maximum* points of F on $\mathcal{B}_{p,\gamma}^{(2)}(k)$.

Remark 2.7. The proof of the regularity of the space $\mathcal{B}_{p,\gamma}^{(2)}(k)$ presented in Proposition 2.1 does not apply to the space $\mathcal{B}_{p,\gamma}^{(1)}(k)$; more precisely, the failure of the proof is in the existence of the vector field Z satisfying (2.28). Observe indeed that, in the case of $\mathcal{B}_{p,\gamma}^{(1)}(k)$, the derivative $\dot{\sigma}$ is only defined as an L^2 -function, and in general it is not a continuous curve.

This fact is the reason why we have to introduce here our global variational setup using the space $\mathcal{B}_{p,\gamma}^{(2)}(k)$.

However, the same proof of Proposition 2.1 can be adapted to prove that, if σ is C^1 , then a suitable neighborhood of σ in $\mathcal{B}_{p,\gamma}^{(1)}(k)$ has the structure of a smooth Hilbert manifold. The proof is more delicate; we omit the details that can be found in a forthcoming paper (see [8]).

As a matter of facts, we will see that the solutions to our variational problem as given in Section 4 are indeed smooth curves (see Proposition 4.1). This fact will allow us to work in the space $\mathcal{B}_{p,\gamma}^{(1)}(k)$ in the second part of the paper (starting from Section 4), when we will be studying the *local* properties of the brachistochrones, i.e., the properties of objects that are defined only around the brachistochrones, like for instance the second variation of T , the Jacobi fields, conjugate points and the Morse Index Theorem for brachistochrones (Theorem 7.12).

We summarize the main properties of the set $\mathcal{B}_{p,\gamma}^{(1)}(k)$ as follows:

Proposition 2.8. $\mathcal{B}_{p,\gamma}^{(1)}(k)$ is a metric space with the metric induced by $H^1([0, 1], \mathcal{M})$. The inclusion $\iota : \mathcal{B}_{p,\gamma}^{(2)}(k) \hookrightarrow \mathcal{B}_{p,\gamma}^{(1)}(k)$ is continuous and it has dense image.

If $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ is a map of class C^1 , then there exists a neighborhood \mathcal{V}_σ of σ in $\mathcal{B}_{p,\gamma}^{(1)}(k)$ that has the structure of an infinite dimensional Hilbertian manifold. In particular, $\mathcal{B}_{p,\gamma}^{(1)}(k)$ has a dense open subset that is a smooth Hilbert manifold.

If $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ is a curve of class C^1 , then, for all $\sigma_1 \in \mathcal{V}_\sigma$, the tangent space $T_{\sigma_1} \mathcal{V}_\sigma$ can be identified with the Hilbert subspace of $T_{\sigma_1} \Omega_{p,\gamma}^{(1)}$ given by:

$$(2.44) \quad T_{\sigma_1} \mathcal{V}_\sigma = \left\{ \zeta \in T_{\sigma_1} \Omega_{p,\gamma}^{(1)} : \exists C_\zeta \in \mathbb{R} \text{ such that } \langle \nabla_{\dot{\sigma}} \zeta, Y \rangle - \langle \zeta, \nabla_{\dot{\sigma}} Y \rangle \equiv C_\zeta \text{ and } \langle \nabla_{\dot{\sigma}} \zeta, \dot{\sigma} \rangle \equiv \frac{T_\sigma C_\zeta}{k} \right\}.$$

The restriction of the travel time functional T to each neighborhood of the form \mathcal{V}_σ , for some $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ of class C^1 , is smooth, and the same result of Corollary 2.2 holds.

Proof. Convergence in the space $\mathcal{B}_{p,\gamma}^{(2)}(k)$ clearly implies the convergence in $\mathcal{B}_{p,\gamma}^{(1)}(k)$, which implies that the inclusion $\iota : \mathcal{B}_{p,\gamma}^{(2)}(k) \hookrightarrow \mathcal{B}_{p,\gamma}^{(1)}(k)$ is continuous.

For the second part of the thesis, it suffices to adapt the proof of Proposition 2.1, and the details will be omitted. \square

We can give the following definition:

Definition 2.9. A curve $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ is said to be a *regular* point of $\mathcal{B}_{p,\gamma}^{(1)}(k)$ if $\mathcal{B}_{p,\gamma}^{(1)}(k)$ has the structure of a smooth Hilbert manifold in a neighborhood \mathcal{V}_σ of σ . By Proposition 2.8, every curve $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ of class C^1 is a regular point of $\mathcal{B}_{p,\gamma}^{(1)}(k)$. A *critical point* of T in $\mathcal{B}_{p,\gamma}^{(1)}(k)$ is a regular point σ of $\mathcal{B}_{p,\gamma}^{(1)}(k)$ such that $dT(\sigma) = 0$ in $T_\sigma \mathcal{V}_\sigma$.

To conclude this section, we remark that, in perfect analogy with Proposition 2.8, if σ is a regular point in $\mathcal{B}_p^{(1)}(k)$ or in $\mathcal{B}^{(1)}(k)$, then these two sets have the structure of smooth manifolds around σ . Their tangent spaces are given by:

$$(2.45) \quad T_\sigma \mathcal{B}_p^{(1)}(k) = \left\{ \zeta \in T_\sigma H^1([0, 1], U_k) : \zeta(0) = 0, \exists C_\zeta \in \mathbb{R} \text{ such that} \right. \\ \left. \langle \nabla_{\dot{\sigma}} \zeta, Y \rangle - \langle \zeta, \nabla_{\dot{\sigma}} Y \rangle \equiv C_\zeta \text{ and } \langle \nabla_{\dot{\sigma}} \zeta, \dot{\sigma} \rangle \equiv \frac{T_\sigma C_\zeta}{k} \right\}.$$

and

$$(2.46) \quad T_\sigma \mathcal{B}^{(1)}(k) = \left\{ \zeta \in T_\sigma H^1([0, 1], U_k) : \exists C_\zeta \in \mathbb{R} \text{ such that} \right. \\ \left. \langle \nabla_{\dot{\sigma}} \zeta, Y \rangle - \langle \zeta, \nabla_{\dot{\sigma}} Y \rangle \equiv C_\zeta \text{ and } \langle \nabla_{\dot{\sigma}} \zeta, \dot{\sigma} \rangle \equiv \frac{T_\sigma C_\zeta}{k} \right\}.$$

3. AN ABSTRACT APPROACH TO THE LAGRANGE MULTIPLIER METHOD. THE FIRST VARIATION OF THE TRAVEL TIME

In this Section we use the Lagrange multiplier technique to derive a system of differential equation satisfied by the brachistochrones, and to extend the variational principle proven in [9].

In order to use this technique, we need a global Banach differentiable structure on our set of maps, and for this reason we will work in the space $\mathcal{B}_{p,\gamma}^{(2)}(k)$ rather than $\mathcal{B}_{p,\gamma}^{(1)}(k)$ (see Remark 2.7 and Proposition 2.8). This approach has the unpleasant drawback of making our notations and calculations much heavier then one would expect. This is due to the fact that the duality in the Sobolev spaces H^1 and \tilde{H}^1 , which are the natural images for the map \mathcal{F} defined by (2.19), involves also products of the first derivatives of the maps and of the Lagrange multipliers, resulting in very lengthy formulas that make it a complicated task to determine an explicit form of the Euler–Lagrange equation satisfied by the critical points of our functional.

To overcome this difficulty, the authors have decided to use the formalism of *generalized functions* and *distributions* on Sobolev spaces, which will make the computations formally similar to the *naive* calculations made by the classical variationalists of the last century. Unfortunately, our problem does not fit perfectly in the theory of distributions on Sobolev spaces presented in standard textbooks, and we are forced to develop our own theory from scratch. Hopefully, the formalism developed here will be adaptable to the study of other variational problems involving several constraints and requiring a *high* degree of regularity for the trial maps.

The first part of this section is devoted to this aim, and it is of rather technical nature. A first time reader who wants to avoid technicalities and who is willing to make an act of faith, after reading formula (3.2) can just skip everything that comes before formula (3.10) without seriously jeopardizing his/her general understanding of the subject.

Keeping Remark 2.6 in mind, in the notation of Section 2 (recall in particular formulas (2.20), (2.22) and (2.23), we want to extremize the action functional $F(\sigma) = \frac{1}{2} \int_0^1 \langle \dot{\sigma}, \dot{\sigma} \rangle ds$ in the space of all curves $\sigma \in \Omega_{p,\gamma}^{(2)}$ subject to the constraint $\mathcal{F}(\sigma) \in \mathcal{C}^- \times \{0\}$.

Then, $\sigma \in \mathcal{B}_{p,\gamma}^{(2)}(k)$ is a solution to our variational problem if and only if there exists an element Λ in the dual space of $\tilde{H}^1([0, 1], \mathbb{R}) \times H^1([0, 1], \mathbb{R})$ such that

$$(3.1) \quad dF(\sigma) - \Lambda \circ (\Pi \circ d\mathcal{F}(\sigma)) : T_\sigma \Omega_{p,\gamma}^{(2)} \mapsto \mathbb{R}$$

vanishes identically. In this case, Λ is unique, and it is the *Lagrange multiplier* of σ ; from a physical point of view, Λ represents the constraint forces acting on the particle moving along σ .

A Lagrangian multiplier for our variational problem is of the form $\Lambda = (\lambda, \mu)$, where $\lambda \in \tilde{H}^1([0, 1], \mathbb{R})^*$ and $\mu \in H^1([0, 1], \mathbb{R})^*$; here the $*$ means the dual space in the sense of Banach spaces. Observe that $\tilde{H}^1([0, 1], \mathbb{R})^*$ can be identified with the closed subspace of $H^1([0, 1], \mathbb{R})^*$ consisting of functionals vanishing on constant functions.

It is convenient to write the duality in the spaces $H^1([0, 1], \mathbb{R})$ and $\tilde{H}^1([0, 1], \mathbb{R})$ in the form:

$$(3.2) \quad \lambda(f) = \int \lambda f \quad \text{and} \quad \mu(f) = \int \mu f,$$

where λ and μ are seen as *generalized functions*. Indeed, we know that not all the continuous functionals on the spaces H^1 and \tilde{H}^1 are of the form (3.2) for some function λ and μ .

This argument introduces a substantial simplification in the computations involving dual spaces, but it needs a concrete formalization, which is rather technical and it is done once and for all in the following.

We introduce the following formalism. Let I denote the interval $[0, 1]$. For each $i \in \mathbb{N}$, let D^i be the dual space $H^i(I, \mathbb{R})^*$. If $\pi_E : E \mapsto \mathcal{M}$ is any fiber bundle over \mathcal{M} with projection π_E , given $\sigma \in H^i(I, \mathcal{M})$, let $H^i(I, \sigma, E)$ denote the set of maps $\omega \in H^i(I, E)$ such that $\pi_E \circ \omega = \sigma$. We will consider in particular the tangent bundle $T\mathcal{M}$ and the cotangent bundle $T\mathcal{M}^*$ with their canonical projections onto \mathcal{M} .

Finally, we denote by D_σ^i the dual space $H^i(I, \sigma, T\mathcal{M}^*)^*$.

We remark that there are canonical inclusions $D^i \subset D^{i+1}$ and $D_\sigma^i \subset D_\sigma^{i+1}$ given by restriction of the linear functionals. By convention, keeping in mind the Riesz representation theorem for Hilbert spaces, we define $H^0 = D^0 = L^2(I, \mathbb{R})$ and $D_\sigma^0 = L^2(I, \sigma, T\mathcal{M})$.

We consider the following operations in the spaces D^i , D_σ^i and H^i :

- (a) For $\lambda \in D^i$, $i \geq 1$, and $f \in H^i(I, \mathbb{R})$, $(\lambda f) \in D^i$ is defined by $(\lambda f)(\phi) = \lambda(f\phi)$. Observe that this product is well defined because, $H^i(I, \mathbb{R})$ is an algebra (i.e., closed with respect to products) and the product $(f, \phi) \mapsto f\phi$ is continuous in $H^i(I, \mathbb{R})$. The validity of the operations defined in the other items is checked by similar arguments.
- (b) For $\lambda \in D^i$, $i \geq 1$, and $V \in H^i(I, \sigma, T\mathcal{M})$, $(\lambda V) \in D_\sigma^i$ is defined by $(\lambda V)(\alpha) = \lambda(\alpha(V))$, for $\alpha \in H^i(I, \sigma, T\mathcal{M}^*)$.
- (c) For $f \in H^i(I, \mathbb{R})$, $i \geq 1$, and $\nu \in D_\sigma^i$, $(f\nu) \in D_\sigma^i$ is defined by $(f\nu)(\alpha) = \nu(f\alpha)$, for $\alpha \in H^i(I, \sigma, T\mathcal{M}^*)$.
- (d) For $V \in H^i(I, \sigma, T\mathcal{M})$ and $\nu \in D_\sigma^i$, $i \geq 1$, the inner product $\langle \nu, V \rangle \equiv \langle V, \nu \rangle \in D^i$ is defined by $\langle \nu, V \rangle(\phi) = \nu(\langle \phi V, \cdot \rangle)$ for $\phi \in H^i(I, \mathbb{R})$.
- (e) For $\lambda \in D^i$, $i \geq 0$, we define $\int_I \lambda = \lambda(1) \in \mathbb{R}$.
- (f) For $\lambda \in D^i$, $i \geq 0$, we denote by $\tilde{\lambda}$ the element in D^{i+1} defined by $\tilde{\lambda}(\phi) = -\lambda(\phi')$ for $\phi \in H^{i+1}(I, \mathbb{R})$. Observe that $\tilde{\lambda}$ is well defined because the map $\phi \mapsto \phi'$ from $H^{i+1}(I, \mathbb{R})$ to $H^i(I, \mathbb{R})$ is linear and continuous. Note also that λ is a sort of *distributional derivative*, but keeping in mind that λ is an element of a dual space of functions which *do not* vanish on the boundary. In particular, even for differentiable functions λ , it is not true that $\tilde{\lambda} = \lambda'$. An explicit form of $\tilde{\lambda}$ is given in part 10 of Proposition 3.1.
- (g) For $\nu \in D_\sigma^i$, $i \geq 0$, the element $\tilde{\nu} \in D_\sigma^{i+1}$ is defined by $\tilde{\nu}(\alpha) = -\nu(\nabla_\sigma \alpha)$, where α belongs to $H^{i+1}(I, \sigma, T\mathcal{M}^*)$ and $\nabla_\sigma \alpha$ is the covariant derivative of the covector α along σ . This means that, if α is the covector given by $\langle V, \cdot \rangle$ for some $V \in H^{i+1}(I, \sigma, T\mathcal{M})$, then

$\nabla_{\dot{\sigma}}\alpha = \langle \nabla_{\dot{\sigma}}V, \cdot \rangle$. The element $\tilde{\nu}$ is the distributional derivative for covectors, analogue to formula (f) (see part 11 of Proposition 3.1).

For $t_0 \in I$, we denote by $\delta_{t_0} \in D^1$ the *Dirac delta* at t_0 , which is the element defined by $\delta_{t_0}(\phi) = \phi(t_0)$ for all $\phi \in H^i(I, \mathbb{R})$; moreover, for $A \in T_{\sigma(t_0)}\mathcal{M}$, $\delta_{t_0}^A \in D_{\sigma}^i$ will denote the element defined by $\delta_{t_0}^A(\alpha) = \alpha(t_0)(A)$.

For $V \in H^i(I, \sigma, T\mathcal{M})$, $t_0 \in I$ and $A \in T_{\sigma(t_0)}\mathcal{M}$, we have:

$$(3.3) \quad \langle V, \delta_{t_0}^A \rangle = \langle V(t_0), A \rangle \delta_{t_0}.$$

Namely, using property (d) above, for $\phi \in H^i(I, \mathbb{R})$ we have:

$$\begin{aligned} \langle V, \delta_{t_0}^A \rangle(\phi) &= \delta_{t_0}^A(\langle \phi V, \cdot \rangle) = \langle \phi(t_0) V(t_0), A \rangle = \phi(t_0) \langle V(t_0), A \rangle = \\ &= \langle V(t_0), A \rangle \cdot \delta_{t_0}(\phi). \end{aligned}$$

Proposition 3.1. *The following statements hold true:*

1. for $\lambda \in D^i$ and $\phi \in H^i(I, \mathbb{R})$, $i \geq 1$, it is $\lambda(\phi) = \int_I \lambda \phi$;
2. the dual space $\tilde{H}^1(I, \mathbb{R})^*$ is identified with the closed subspace of D^1 consisting of elements λ satisfying $\int_I (\lambda \cdot 1) = 0$;
3. if $\tilde{\lambda} = 0$, then $\lambda = 0$;
4. for $\nu \in D_{\sigma}^i$ and $V \in H^{i+1}(I, \sigma, T\mathcal{M})$, $i \geq 0$, it is $\int_I \langle \nu, \nabla_{\dot{\sigma}} V \rangle = - \int_I \langle \tilde{\nu}, V \rangle$;
5. there exists a continuous linear injection of $L^1(I, \mathbb{R})$ into D^i , $i \geq 1$, given by the map $\lambda \in L^1(I, \mathbb{R}) \mapsto \hat{\lambda} \in D^i$, where $\hat{\lambda}(\phi) = \int_I \lambda(s) \phi(s) ds$ for all $\phi \in H^i(I, \mathbb{R})$;
6. if $L^1(I, \sigma, T\mathcal{M})$ denotes the set of vector fields along σ whose Riemannian length (2.2) is Lebesgue integrable, then there is a continuous linear injection of $L^1(I, \sigma, T\mathcal{M})$ into D_{σ}^i , $i \geq 1$, given by $\nu \in L^1(I, \sigma, T\mathcal{M}) \mapsto \hat{\nu} \in D_{\sigma}^i$, where $\hat{\nu}(\alpha) = \int_I \alpha(t) \nu(t) dt$ for $\alpha \in H^i(I, \sigma, T\mathcal{M}^*)$;
7. if $\psi \in D^1$ is such that $\tilde{\psi} \in D^2$ is also in D^1 (recall the inclusion $D^1 \subset D^2$), then $\psi \in L^2(I, \mathbb{R})$; similarly, if $\psi, \tilde{\psi} \in D_{\sigma}^1$, then $\psi \in L^2(I, \sigma, T\mathcal{M})$;
8. for $\lambda \in D^i$ and $f \in H^{i+1}(I, \mathbb{R}) \subset H^i(I, \mathbb{R})$, $i \geq 0$, it is $(\lambda f) = \tilde{\lambda} f + \lambda f'$;
9. for $\lambda \in D^i$ and $V \in H^{i+1}(I, \sigma, T\mathcal{M}) \subset H^i(I, \sigma, T\mathcal{M})$, $i \geq 0$, it is $(\lambda V) = \tilde{\lambda} V + \lambda \nabla_{\dot{\sigma}} V$;
10. for $f \in H^1(I, \mathbb{R})$, it is $\tilde{f} = f(0) \delta_0 - f(1) \delta_1 + f'$;
11. for $V \in H^1(I, \sigma, T\mathcal{M})$, it is $\tilde{V} = \delta_0^{V(0)} - \delta_1^{V(1)} + \nabla_{\dot{\sigma}} V$.

Proof. For part 1, it is $\lambda(\phi) = \lambda(\phi \cdot 1)$. By (a), it is $\lambda(\phi \cdot 1) = (\lambda \phi)(1)$ and by (e) $(\lambda \phi)(1) = \int_I \lambda \phi$.

Part 2 is simply the fact that elements in the dual space of $\tilde{H}^1(I, \mathbb{R})$ are characterized by the property of vanishing on constant functions.

For part 3, it suffices to observe that the map $\phi \mapsto \phi'$ is surjective from $H^{i+1}(I, \mathbb{R})$ to $H^i(I, \mathbb{R})$.

For part 4, using (d) and (e), we have:

$$\int_I \langle \nu, \nabla_{\dot{\sigma}} V \rangle = \langle \nu, \nabla_{\dot{\sigma}} V \rangle(1) = \nu(\langle 1 \cdot \nabla_{\dot{\sigma}} V, \cdot \rangle).$$

On the other hand, by (e) and (g) we have:

$$\int_I \langle \tilde{\nu}, V \rangle = \langle \tilde{\nu}, V \rangle(1) = \tilde{\nu}(\langle 1 \cdot V, \cdot \rangle) = -\nu(\langle V, \cdot \rangle') = -\nu(\langle \nabla_{\dot{\sigma}} V, \cdot \rangle),$$

which proves the claim.

For part 5, observe that $\hat{\lambda}$ is a well defined element in the dual of $H^i(I, \mathbb{R})$. The linearity of the map $\lambda \mapsto \hat{\lambda}$ is trivial; the continuity depends on the fact that convergence in H^1 implies uniform convergence. Finally, the injectivity is simply the Fundamental Theorem of Calculus of Variations.

Part 6 is proven analogously. Namely, using an orthonormal frame along σ , one reduces the problem to the case $\mathcal{M} = \mathbb{R}^m$. In this case the proof of part 5 can be repeated *verbatim* for each component of ν .

Using part 5 and 6, we will identify each $\lambda \in L^1(I, \mathbb{R})$ with $\hat{\lambda} \in D^i$ and every $V \in L^1(I, \sigma, T\mathcal{M})$ with $\hat{V} \in D_\sigma^i$. Suppressing the symbol $\hat{\cdot}$, for all $\lambda \in L^1(I, \mathbb{R})$, $V \in L^1(I, \sigma, T\mathcal{M})$, $f \in H^i(I, \mathbb{R})$ and $\alpha \in H^i(I, \sigma, T\mathcal{M}^*)$ we will write concisely:

$$(3.4) \quad \lambda(f) = \int_I \lambda(t) f(t), \quad \text{and} \quad V(\alpha) = \int_I \alpha(t) V(t).$$

To prove part 7, observe that the map $f \mapsto f'$ from $H^1(I, \mathbb{R})$ to $L^2(I, \mathbb{R})$ is continuous and surjective. Hence, if $\tilde{\psi} \in D^1$, then the map $f' \mapsto \int_I \psi f' = -\int_I \tilde{\psi} f$, where f is the unique primitive of f' such that $f(0) = 0$, gives a continuous linear functional on $L^2(I, \mathbb{R})$, and the conclusion follows by Riesz Theorem. The second half is proven similarly.

The formulas in part 8 and 9 are the product rules for the \sim -derivative. We prove 9 as follows. For $\alpha \in H^{i+1}(I, \sigma, T\mathcal{M}^*)$, we have:

$$(\widetilde{\lambda V})(\alpha) = -(\lambda V)(\nabla_{\dot{\sigma}} \alpha) = -\lambda(\nabla_{\dot{\sigma}} \alpha(V)).$$

On the other hand, we compute:

$$\begin{aligned} (\tilde{\lambda} V + \lambda \nabla_{\dot{\sigma}} V)(\alpha) &= (\tilde{\lambda} V)(\alpha) + (\lambda \nabla_{\dot{\sigma}} V)(\alpha) = \tilde{\lambda}(\alpha(V)) + \lambda(\alpha(\nabla_{\dot{\sigma}} V)) = \\ &= -\lambda(\alpha(V)') + \lambda(\alpha(\nabla_{\dot{\sigma}} V)) = \\ &= -\lambda(\nabla_{\dot{\sigma}} \alpha(V)) - \lambda(\alpha(\nabla_{\dot{\sigma}} V)) + \lambda(\alpha(\nabla_{\dot{\sigma}} V)) = -\lambda(\nabla_{\dot{\sigma}} \alpha(V)). \end{aligned}$$

Part 8 is proven similarly.

We omit the proof of part 10 and we prove part 11. For $\alpha \in H^{i+1}(I, \sigma, T\mathcal{M}^*)$, using (3.4), we have:

$$\begin{aligned} \tilde{V}(\alpha) &= -V(\nabla_{\dot{\sigma}} \alpha) = -\int_I (\nabla_{\dot{\sigma}} \alpha(t)) V(t) = -\int_I [(\alpha(t)V(t))' - \alpha(t)\nabla_{\dot{\sigma}} V(t)] = \\ &= -\alpha(t)V(t) \Big|_0^1 + \int_I \alpha(t)\nabla_{\dot{\sigma}} V(t) = \alpha(0)(V(0)) - \alpha(1)(V(1)) + \nabla_{\dot{\sigma}} V(\alpha) = \\ &= \delta_0^{V(0)}(\alpha) - \delta_1^{V(1)}(\alpha) + \nabla_{\dot{\sigma}} V(\alpha). \end{aligned}$$

This concludes the proof of Proposition 3.1. \square

We now present three preliminary results that will be needed in the computation of the first variation for the travel time functional:

Lemma 3.2. *Let $\nu \in D_\sigma^i$ and suppose that $\int_I \langle V, \nu \rangle = 0$ for all $V \in H^i(I, \sigma, T\mathcal{M})$ such that $V(0) = 0$ and $V(1)$ is parallel to $Y(\sigma(1))$. Then, we have $\nu = \delta_0^A + \delta_1^B$ for some $A \in T_{\sigma(0)}\mathcal{M}$ and $B \in T_{\sigma(1)}\mathcal{M}$ with $\langle B, Y(\sigma(1)) \rangle = 0$.*

Proof. Under the hypotheses, it is $\nu(\alpha) = 0$ for all $\alpha \in H^i(I, \sigma, T\mathcal{M}^*)$ such that $\alpha(0) = 0$ and such that $\alpha(1)$ is a multiple of the covector $\langle Y(\sigma(1)), \cdot \rangle$. The subspace H of such α 's has codimension equal to $(2m-1)$ in $H^i(I, \sigma, T\mathcal{M}^*)$. Then, the annihilator H° of H in D_σ^i has dimension $(2m-1)$. The subspace N of D_σ^i consisting of elements ν of the form $\delta_0^A + \delta_1^B$ for some $A \in T_{\sigma(0)}\mathcal{M}$ and $B \in T_{\sigma(1)}\mathcal{M}$ with $\langle B, Y(\sigma(1)) \rangle = 0$ clearly has dimension $(2m-1)$ and it is contained in the annihilator of H . Thus, $N = H^\circ$ and we are done. \square

Lemma 3.3. *Let $\lambda \in D^1$ be fixed. If $\tilde{\lambda} = c_0 \delta_0 + c_1 \delta_1$ for some $c_0, c_1 \in \mathbb{R}$, then necessarily $c_0 = -c_1$ and $\lambda \equiv c_0$ is constant, i.e., $\lambda(\phi) = \int_I c_0 \phi(t) dt$ for all $\phi \in H^1(I, \mathbb{R})$.*

Proof. First of all, observe that there exists no $\lambda \in D^1$ such that $\tilde{\lambda} = \delta_0$. Namely, if $\tilde{\lambda} = \delta_0$ and $\phi \in H^1(I, \mathbb{R})$, then it would be $\tilde{\lambda}(\phi) = \phi(0)$, and so $\lambda(\phi') = -\phi(0)$. On the other hand, for all constants $c \in \mathbb{R}$, it would be $\tilde{\lambda}(\phi + c) = \phi(0) + c$, and $\tilde{\lambda}(\phi + c) = -\lambda(\phi') = -\phi(0)$, which is a contradiction.

It follows that there exists no $\lambda \in D^1$ such that $\tilde{\lambda} = c_0 \delta_0 + c_1 \delta_1$ with $c_0 \neq -c_1$. Indeed, if such λ existed, then the element $\lambda_1 = (c_0 + c_1)^{-1}(\lambda + c_1)$ would satisfy $\tilde{\lambda}_1 = \delta_0$.

Finally, suppose that $\tilde{\lambda} = c_0 \delta_0 - c_0 \delta_1$. Then, $(\lambda - c_0) = 0$, and by part 3 of Proposition 3.1, $\lambda \equiv c_0$. \square

The following simple result states the well known fact that the Dirac delta's are not given by any L^1 -function:

Lemma 3.4. *If $\lambda \in L^1(I, \mathbb{R})$ is such that $\hat{\lambda} \in D^i$ is of the form $c_0 \delta_0 + c_1 \delta_1$ for some $c_0, c_1 \in \mathbb{R}$, then $\lambda \equiv 0$ and $c_0 = c_1 = 0$. Similarly, if $\nu \in L^1(I, \sigma, T\mathcal{M})$ is such that $\hat{\nu} \in D_\sigma^i$ is of the form $\delta_0^A + \delta_0^B$ for some vectors $A \in T_{\sigma(0)}\mathcal{M}$ and $B \in T_{\sigma(1)}\mathcal{M}$, then $\nu \equiv 0$ and $A = B = 0$.*

Proof. If $\hat{\lambda} = c_0 \delta_0 + c_1 \delta_1$, then $\int_I \lambda(t) \phi(t) dt = 0$ for all smooth function ϕ with support contained in $]0, 1[$. This implies $\lambda \equiv 0$. The proof of the second part of the Lemma is analogous. \square

We are now ready to determine the Euler–Lagrange equation for the critical points of the travel time functional in $\mathcal{B}_{p,\gamma}^{(2)}(k)$. Recalling (2.21), (3.1) and part 2 of Proposition 3.1, we now fix a curve $\sigma \in \mathcal{B}_{p,\gamma}^{(2)}(k)$. Recall from the definition (2.16) of $\mathcal{B}_{p,\gamma}^{(2)}(k)$ that there exists $\mathcal{T}_\sigma > 0$ such that:

$$(3.5) \quad \langle \dot{\sigma}, Y \rangle \equiv -k \mathcal{T}_\sigma, \quad \text{and} \quad \langle \dot{\sigma}, \dot{\sigma} \rangle = -\mathcal{T}_\sigma^2.$$

We assume that there exist $\lambda, \mu \in D^1$ (see part 2 of Proposition 3.1), with $\int_I \lambda = 0$, such that the equation:

$$(3.6) \quad 0 = \int_I \langle \nabla_{\dot{\sigma}} V, \dot{\sigma} \rangle ds - \lambda \left(\langle \nabla_{\dot{\sigma}} V, Y \rangle - \langle V, \nabla_{\dot{\sigma}} Y \rangle \right) + \\ - \mu \left(2 \langle \dot{\sigma}, Y \rangle (\langle \nabla_{\dot{\sigma}} V, Y \rangle - \langle V, \nabla_{\dot{\sigma}} Y \rangle) + 2k^2 \langle \nabla_{\dot{\sigma}} V, \dot{\sigma} \rangle \right)$$

is satisfied for all $V \in T_{\sigma} \Omega_{p,\gamma}^{(2)}$. Using the formalism introduced in the first part of the Section, we rewrite equation (3.6) as:

$$(3.7) \quad 0 = \int_I \langle V, \lambda \nabla_{\dot{\sigma}} Y + 2\mu \langle \dot{\sigma}, Y \rangle \nabla_{\dot{\sigma}} Y \rangle +$$

$$(3.8) \quad + \int_I \langle \nabla_{\dot{\sigma}} V, \dot{\sigma} - \lambda Y - 2\mu \langle \dot{\sigma}, Y \rangle Y - 2\mu k^2 \dot{\sigma} \rangle.$$

In the above formula, the "products" between the dual maps λ and μ with functions or vector fields along σ have to be interpreted in the sense of the operations (a)–(c) above; moreover, the inner product $\langle \cdot, \cdot \rangle$ in (3.7) and (3.8) is meant in the sense of (d).

Observe that the elements:

$$(3.9) \quad \phi = \lambda \nabla_{\dot{\sigma}} Y + 2\mu \langle \dot{\sigma}, Y \rangle \nabla_{\dot{\sigma}} Y, \quad \text{and} \quad \psi = \dot{\sigma} - \lambda Y - 2\mu \langle \dot{\sigma}, Y \rangle Y - 2\mu k^2 \dot{\sigma}$$

are in D_σ^1 .

We need the following *regularity* result for the Lagrangian multipliers λ and μ :

Lemma 3.5. *The Lagrangian multipliers λ and μ are indeed L^2 -functions, i.e., there exist $f_\lambda, f_\mu \in L^2(I, \mathbb{R}) \subset L^1(I, \mathbb{R})$ such that $\lambda = \hat{f}_\lambda$ and $\mu = \hat{f}_\mu$.*

Proof. From (3.7), (3.8) and (3.9), we have $\int \langle V, \phi \rangle + \int \langle \nabla_{\dot{\sigma}} V, \psi \rangle = \int \langle V, \phi - \tilde{\psi} \rangle = 0$ for all $V \in H^2(I, \sigma, T\mathcal{M})$ such that $V(0) = 0$ and $V(1)$ is parallel to $Y(\sigma(1))$. From Lemma 3.2 it follows that $\phi - \tilde{\psi}$ is a linear combination of delta's, and in particular, $\phi - \tilde{\psi}$ is in D_σ^1 . Hence, $\tilde{\psi}$ is in D_σ^1 , and, by part 7 of Proposition 3.1, $\psi \in L^2(I, \sigma, T\mathcal{M})$. Since $\dot{\sigma} \in H^1(I, TM)$, then $\langle \psi, \dot{\sigma} \rangle$ is in $L^2(I, \mathbb{R})$; computing explicitly, we have:

$$\langle \psi, \dot{\sigma} \rangle = -\mathcal{T}_\sigma^2 + \lambda k \mathcal{T}_\sigma - 2\mu k^2 \mathcal{T}_\sigma^2 + 2\mu k^2 \mathcal{T}_\sigma^2 = -\mathcal{T}_\sigma^2 + \lambda k \mathcal{T}_\sigma \in L^2(I, \mathbb{R}),$$

hence $\lambda \in L^2(I, \mathbb{R})$. Then, from the definition (3.9) of ψ , we obtain that $\mu \mathcal{T}_\sigma Y - \mu k \dot{\sigma} \in L^2(I, T\mathcal{M})$; multiplying by Y we have:

$$\mu \mathcal{T}_\sigma \langle Y, Y \rangle + \mu k^2 \mathcal{T}_\sigma = \mu \mathcal{T}_\sigma (\langle Y, Y \rangle + k^2) \in L^2(I, \mathbb{R}).$$

Since $(\langle Y, Y \rangle + k^2)^{-1} \in L^\infty(I, \mathbb{R})$ (because σ has image in U_k), it follows that $\mu \in L^2(I, \mathbb{R})$ and the proof is concluded. \square

We use the operation (f) to "integrate by parts" (3.8), and, keeping in mind parts 8 and 9 of Proposition 3.1, we obtain

$$\begin{aligned} 0 &= \int_I \langle V, \lambda \nabla_{\dot{\sigma}} Y + 2\mu \langle \dot{\sigma}, Y \rangle \nabla_{\dot{\sigma}} Y \rangle + \\ (3.10) \quad &- \int_I \langle V, \tilde{\sigma} - \tilde{\lambda} Y - \lambda \nabla_{\dot{\sigma}} Y - 2\tilde{\mu} \langle \dot{\sigma}, Y \rangle Y - 2\mu \langle \dot{\sigma}, Y \rangle \nabla_{\dot{\sigma}} Y \rangle + \\ &+ \int_I \langle V, 2\tilde{\mu} k^2 \dot{\sigma} + 2\mu k^2 \nabla_{\dot{\sigma}} \dot{\sigma} \rangle, \end{aligned}$$

for all $V \in T_{\sigma} \Omega_{p, \gamma}^{(2)}$. Observe that, when using parts 8 and 9 of Proposition 3.1, if the functions involved are only in H^1 (like in this particular case the function $\dot{\sigma}$) then they must be multiplied by distributions in $D^0 = L^2$ for the rule to apply. This is where we use Lemma 3.5.

We substitute

$$\tilde{\sigma} = \nabla_{\dot{\sigma}} \dot{\sigma} + \delta_0^{\dot{\sigma}(0)} - \delta_1^{\dot{\sigma}(1)}$$

in (3.10), and, from Lemma 3.2, we have:

$$\begin{aligned} &\lambda \nabla_{\dot{\sigma}} Y + 4\mu \langle \dot{\sigma}, Y \rangle \nabla_{\dot{\sigma}} Y - \nabla_{\dot{\sigma}} \dot{\sigma} + \tilde{\lambda} Y + \lambda \nabla_{\dot{\sigma}} Y + 2\tilde{\mu} \langle \dot{\sigma}, Y \rangle Y + \\ (3.11) \quad &+ 2\tilde{\mu} k^2 \dot{\sigma} + 2\mu k^2 \nabla_{\dot{\sigma}} \dot{\sigma} = \delta_0^A + \delta_0^{\dot{\sigma}(0)} + \delta_1^B - \delta_1^{\dot{\sigma}(1)}, \end{aligned}$$

for some $A \in T_{\sigma(0)} \mathcal{M}$ and some $B \in T_{\sigma(1)} \mathcal{M}$ such that $\langle B, Y(\sigma(1)) \rangle = 0$.

Now, we multiply equation (3.11) by $\dot{\sigma}$, and since $\langle \nabla_{\dot{\sigma}} Y, \dot{\sigma} \rangle = \langle \nabla_{\dot{\sigma}} \dot{\sigma}, \dot{\sigma} \rangle = 0$ and $\langle \dot{\sigma}, Y \rangle = -k \mathcal{T}_\sigma$, $\langle \dot{\sigma}, \dot{\sigma} \rangle = -\mathcal{T}_\sigma^2$, using (3.3), we get:

$$(3.12) \quad \tilde{\lambda} k \mathcal{T}_\sigma = \langle \dot{\sigma}(0), A \rangle \delta_0 - \mathcal{T}_\sigma^2 \delta_0 + \langle \dot{\sigma}(1), B \rangle \delta_1 + \mathcal{T}_\sigma^2 \delta_1.$$

This means that $\tilde{\lambda}$ is a linear combination of δ_0 and δ_1 . By Lemma 3.3, λ is constant and $\langle \dot{\sigma}(0), A \rangle = -\langle \dot{\sigma}(1), B \rangle$. But λ constant and $\int_I \lambda = 0$ imply immediately:

$$(3.13) \quad \lambda = 0.$$

In particular, we have:

$$(3.14) \quad \langle \dot{\sigma}(0), A \rangle = -\langle \dot{\sigma}(1), B \rangle = \mathcal{T}_\sigma^2.$$

We now substitute $\lambda = \tilde{\lambda} = 0$ in (3.11); multiplying the resulting equation by Y , using (3.14) and recalling that $\langle \dot{\sigma}, Y \rangle$ is constant and that $\langle \nabla_{\dot{\sigma}} \dot{\sigma}, Y \rangle = 0$, we obtain:

$$\begin{aligned} &-4k \mathcal{T}_\sigma \mu \langle \nabla_{\dot{\sigma}} Y, Y \rangle - 2k \mathcal{T}_\sigma \tilde{\mu} \langle Y, Y \rangle - 2k^3 \mathcal{T}_\sigma \tilde{\mu} = \\ &(\langle Y(p), A \rangle - k \mathcal{T}_\sigma) \delta_0 + (\langle Y(\sigma(1)), B \rangle + k \mathcal{T}_\sigma) \delta_1, \end{aligned}$$

which can be written as:

$$(3.15) \quad -2k \mathcal{T}_\sigma (\mu \widetilde{\langle Y, Y \rangle} + k^2 \tilde{\mu}) = (\langle Y(p), A \rangle - k \mathcal{T}_\sigma) \delta_0 + (\langle Y(\sigma(1)), B \rangle + k \mathcal{T}_\sigma) \delta_1.$$

Again, by Lemma 3.3, we have that:

$$(3.16) \quad \langle Y(p), A \rangle = -\langle Y(\sigma(1)), B \rangle = 0,$$

and

$$(3.17) \quad \mu (\langle Y, Y \rangle + k^2) \equiv c$$

for some constant $c \in \mathbb{R}$. Finally, from (3.15) and (3.16) we compute easily $c = \frac{1}{2}$, and

$$(3.18) \quad \mu = \frac{1}{2(k^2 + \langle Y, Y \rangle)}.$$

From (3.18) we compute easily:

$$(3.19) \quad \tilde{\mu} = -\frac{\langle \nabla_{\dot{\sigma}} Y, Y \rangle}{(\langle Y, Y \rangle + k^2)^2} + \mu(0) \delta_0 - \mu(1) \delta_1;$$

substituting (3.5), (3.13), (3.18) and (3.19) into (3.11) gives:

$$(3.20) \quad \begin{aligned} & -\frac{\langle Y, Y \rangle}{\langle Y, Y \rangle + k^2} \nabla_{\dot{\sigma}} \dot{\sigma} - 2k^2 \frac{\langle \nabla_{\dot{\sigma}} Y, Y \rangle}{(\langle Y, Y \rangle + k^2)^2} \dot{\sigma} - \frac{2k \mathcal{T}_{\sigma}}{\langle Y, Y \rangle + k^2} \nabla_{\dot{\sigma}} Y + \\ & + 2k \mathcal{T}_{\sigma} \frac{\langle \nabla_{\dot{\sigma}} Y, Y \rangle}{(\langle Y, Y \rangle + k^2)^2} Y = \\ & = \delta_0^{A+\dot{\sigma}(0)} + \delta_1^{B-\dot{\sigma}(1)} - 2k(-\mathcal{T}_{\sigma} Y(p) \mu(0) + k \dot{\sigma}(0)) \delta_0 + \\ & + 2k(-\mathcal{T}_{\sigma} Y(\sigma(1)) \mu(1) + k \dot{\sigma}(1)) \delta_1. \end{aligned}$$

Observe that for $t_0 \in I$ and $v_0 \in T_{\sigma(t_0)} \mathcal{M}$, it is $v_0 \delta_{t_0} = \delta_{t_0}^{v_0}$, hence, the second member of the equality (3.20) can be written as:

$$\delta_0^{A_1} + \delta_1^{B_1},$$

where

$$\begin{aligned} A_1 &= A + \dot{\sigma}(0) - 2k(-\mathcal{T}_{\sigma} Y(p) \mu(0) + k \dot{\sigma}(0)), \\ B_1 &= B - \dot{\sigma}(1) + 2k(-\mathcal{T}_{\sigma} Y(\sigma(1)) \mu(1) + k \dot{\sigma}(1)). \end{aligned}$$

Hence, by Lemma 3.4, the first member of the equality (3.20) is null, and also $A_1 = B_1 = 0$. Therefore, we obtain the following differential equation for σ :

$$(3.21) \quad \begin{aligned} & \nabla_{\dot{\sigma}} \dot{\sigma} + 2k^2 \frac{\langle \nabla_{\dot{\sigma}} Y, Y \rangle}{\langle Y, Y \rangle (k^2 + \langle Y, Y \rangle)} \dot{\sigma} + \frac{2k \mathcal{T}_{\sigma}}{\langle Y, Y \rangle} \nabla_{\dot{\sigma}} Y + \\ & - 2k \mathcal{T}_{\sigma} \frac{\langle \nabla_{\dot{\sigma}} Y, Y \rangle}{\langle Y, Y \rangle (k^2 + \langle Y, Y \rangle)} Y = 0. \end{aligned}$$

We have proven the following:

Proposition 3.6. *Let $\sigma \in \mathcal{B}_{p,\gamma}^{(2)}(k)$. Then, σ is a brachistochrone of energy k between p and γ if and only if σ is a curve of class C^2 and there exists $\mathcal{T}_{\sigma} > 0$ such that σ satisfies the differential equation (3.21). \square*

Observe that any curve σ in $H^2(I, \mathcal{M})$ that satisfies (3.21) almost everywhere is automatically smooth.

Besides determining the differential equation (3.21), the importance of Proposition 3.6 lies in the fact that, due to the smoothness of the brachistochrones, we will be able to work in the space $\mathcal{B}_{p,\gamma}^{(1)}(k)$ when we are in the vicinity of such a curve (recall Remark 2.7 and Proposition 2.8). This will be done systematically starting from the next Section.

Recalling Definition 2.9, we have the following:

Proposition 3.7. *A curve σ is a brachistochrone of energy k between p and γ if and only if it is a critical point for T in $\mathcal{B}_{p,\gamma}^{(1)}(k)$. \square*

4. THE BRACHISTOCHRONE DIFFERENTIAL EQUATION AND THE FIRST ORDER VARIATIONAL PRINCIPLE REVISITED

In this section we will take a closer look at the differential equation (3.21) and we will prove that it characterizes the brachistochrones between p and γ among all the curves in $\Omega_{p,\gamma}^{(1)}$ satisfying suitable initial conditions.

Proposition 3.6 can be improved as follows:

Proposition 4.1. *A curve $\sigma \in \Omega_{p,\gamma}^{(1)}$ is a brachistochrone of energy k between p and γ if and only if σ is smooth and there exists a $\mathcal{T}_\sigma > 0$ such that σ satisfies (3.21), with initial velocity $\dot{\sigma}(0)$ satisfying:*

$$(4.1) \quad \langle \dot{\sigma}(0), \dot{\sigma}(0) \rangle = -\mathcal{T}_\sigma^2, \quad \text{and} \quad \langle \dot{\sigma}(0), Y(p) \rangle = -k \mathcal{T}_\sigma.$$

Proof. From Proposition 3.6, all we need to prove is that any smooth curve $\sigma \in \Omega_{p,\gamma}^{(1)}$ that satisfies the differential equation (3.21) and whose initial velocity $\dot{\sigma}(0)$ satisfies (4.1) is in $\mathcal{B}_{p,\gamma}^{(1)}(k)$.

To this aim, it suffices to show that the functions $\eta(t) = \langle \dot{\sigma}(t), \dot{\sigma}(t) \rangle + \mathcal{T}_\sigma^2$ and $\theta(t) = \langle \dot{\sigma}(t), Y(\sigma(t)) \rangle + k \mathcal{T}_\sigma$ are constant.

If we multiply (3.21) by Y , we obtain:

$$\langle \nabla_{\dot{\sigma}} \dot{\sigma}, Y \rangle + \frac{2k^2 \langle \nabla_{\dot{\sigma}} Y, Y \rangle}{\langle Y, Y \rangle (k^2 + \langle Y, Y \rangle)} (k \mathcal{T}_\sigma + \langle \dot{\sigma}, Y \rangle) = 0,$$

that can be written as:

$$(4.2) \quad \theta' + u \theta = 0,$$

with

$$u = \frac{2k^2 \langle \nabla_{\dot{\sigma}} Y, Y \rangle}{\langle Y, Y \rangle (k^2 + \langle Y, Y \rangle)}.$$

Since $\theta(0) = 0$, then, the uniqueness of the solution for equation (4.2) implies $\theta \equiv 0$. Now, if we multiply (3.21) by $\dot{\sigma}$, knowing that $\langle \dot{\sigma}, Y \rangle = -k \mathcal{T}_\sigma$ is constant and $\langle \nabla_{\dot{\sigma}} Y, \dot{\sigma} \rangle = 0$, we obtain:

$$\langle \nabla_{\dot{\sigma}} \dot{\sigma}, \dot{\sigma} \rangle + \frac{2k^2 \langle \nabla_{\dot{\sigma}} Y, Y \rangle}{\langle Y, Y \rangle (k^2 + \langle Y, Y \rangle)} (\langle \dot{\sigma}, \dot{\sigma} \rangle + \mathcal{T}_\sigma^2) = 0,$$

that can be written as:

$$(4.3) \quad \frac{1}{2} \eta' + g \eta = 0.$$

Again, since $\eta(0) = 0$, equation (4.3) implies $\eta \equiv 0$ and we are done. \square

We give two more different descriptions of the brachistochrone curves. We first characterize them as curves that minimize *locally* their travel time.

If q is any point in U_k , we denote by γ_q the maximal integral line of Y through q . Moreover, if $I = [a, b] \subseteq [0, 1]$ is any interval, and if q_1, q_2 are any two points in U_k , we define $\mathcal{B}_{q_1, \gamma_{q_2}}^{(1)}(k, I)$ as the space of curves $\tau \in H^1(I, U_k)$ such that $\tau(a) = q_1$, $\tau(b) \in \gamma_{q_2}(\mathbb{R})$, and satisfying $\langle \dot{\tau}, Y \rangle \equiv -k \mathcal{T}_\tau$, $\langle \dot{\tau}, \dot{\tau} \rangle \equiv -\mathcal{T}_\tau^2$ for some $\mathcal{T}_\tau \in \mathbb{R}^+$.

Observe that if $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$, then, for every $I = [a, b] \subseteq [0, 1]$, the restriction of σ to I is a curve in $\mathcal{B}_{\sigma(a), \gamma_{\sigma(b)}}^{(1)}(k, I)$.

Definition 4.2. A curve $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ is said to be a *local minimizer* for the travel time if, for all $0 \leq a < b \leq 1$ such that $b - a$ is sufficiently small, the restriction of σ to the interval $I = [a, b]$ is a minimum point for the travel time functional in the space $\mathcal{B}_{\sigma(a), \gamma_{\sigma(b)}}^{(1)}(k, I)$

Note that Definition 4.2 is essentially the definition of brachistochrones of energy k given in [9]. For curves that are local minimizers of the travel time, the differential equation (3.21) was established in [9] by means of a variational principle, that we can now state in a more complete form.

We denote by Δ the smooth distribution on \mathcal{M} given by the orthocomplement of the vector field Y . Observe that, since Y is timelike, the wrong way Schwartz's inequality implies that Δ is *spacelike*, i.e., the restriction of the Lorentzian metric g on Δ is positive definite.

Let $\psi : \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M}$ be the flow of Y . Recall that, since Y is Killing, then $\psi(\cdot, t)$ is a local isometry for all $t \in \mathbb{R}$; moreover, it is easy to see that the distribution Δ is ψ -invariant, which means that $\psi_x(q, t_0)(\Delta_q) = \Delta_{\psi(q, t_0)}$, where $\psi_x(q, t_0)$ denotes the differential of the map $\psi(\cdot, t_0)$ at the point q . A function $\phi : \mathcal{M} \rightarrow \mathbb{R}$ is said to be Y -invariant if it is constant along the flow lines of Y ; if ϕ is C^1 , this amounts to saying that $\langle Y, \nabla \phi \rangle \equiv 0$.

We define $\Omega_{p, \gamma}^{(1)}(\Delta)$ to be the subset of $\Omega_{p, \gamma}^{(1)}$ consisting of curves with tangent vector at each point lying in Δ :

$$(4.4) \quad \Omega_{p, \gamma}^{(1)}(\Delta) = \left\{ w \in \Omega_{p, \gamma}^{(1)} : \dot{w}(t) \in \Delta_{w(t)}, \forall t \in [0, 1] \right\}.$$

Using the language of sub-Riemannian geometry, we will call *horizontal* the curves in $\Omega_{p, \gamma}^{(1)}(\Delta)$.

By the same arguments of Proposition 2.1, one checks immediately that, since $\langle Y, Y \rangle$ is never vanishing, $\Omega_{p, \gamma}^{(1)}(\Delta)$ is a smooth submanifold of $\Omega_{p, \gamma}^{(1)}$, and that, for $w \in \Omega_{p, \gamma}^{(1)}(\Delta)$, the tangent space $T_w \Omega_{p, \gamma}^{(1)}(\Delta)$ is given by:

$$(4.5) \quad T_w \Omega_{p, \gamma}^{(1)}(\Delta) = \left\{ V \in T_w \Omega_{p, \gamma}^{(1)} : \langle \nabla_{\dot{w}} V, Y \rangle - \langle V, \nabla_{\dot{w}} Y \rangle = 0 \right\}.$$

It will also be useful, as in the case of the spaces $\mathcal{B}_{p, \gamma}^{(2)}(k)$ and $\mathcal{B}_p^{(2)}(k)$ (see formula (2.39), to introduce the spaces $\Omega_p^{(1)}$ and $\Omega_p^{(1)}(\Delta)$, by:

$$(4.6) \quad \Omega_p^{(1)} = \bigcup_{\gamma \subset U_k} \Omega_{p, \gamma}^{(1)}, \quad \text{and} \quad \Omega_p^{(1)}(\Delta) = \bigcup_{\gamma \subset U_k} \Omega_{p, \gamma}^{(1)}(\Delta).$$

We single out the following simple fact:

Lemma 4.3. *Let ϕ be a smooth Y -invariant positive function. Then, the functional*

$$(4.7) \quad E_\phi(w) = \frac{1}{2} \int_0^1 \phi(w) \langle \dot{w}, \dot{w} \rangle_{(R)} dt$$

on $\Omega_{p, \gamma}^{(1)}$ and its restriction to $\Omega_{p, \gamma}^{(1)}(\Delta)$ have the same critical points. These critical points are geodesics in \mathcal{M} with respect to the Riemannian metric $\phi \cdot g_R$ that join p and γ and that are orthogonal to γ .

Proof. The critical points of E_ϕ in $\Omega_{p, \gamma}^{(1)}$ are precisely the geodesics in \mathcal{M} with respect to $\phi \cdot g_R$ that join p and γ and that are orthogonal to γ , i.e., $\langle \dot{w}(1), Y(w(1)) \rangle_{(R)} = 0$. Since ϕ is Y -invariant, then Y is Killing in the metric $\phi \cdot g_R$, thus, for every such geodesic w , the quantity $\langle \dot{w}, Y \rangle_{(R)}$ is constant. Hence $\langle \dot{w}, Y \rangle_{(R)} \equiv 0$ and w is horizontal. Therefore, the critical points of E_ϕ on $\Omega_{p, \gamma}^{(1)}$ belong to $\Omega_{p, \gamma}^{(1)}(\Delta)$, and clearly they are critical points of the restriction of E_ϕ to $\Omega_{p, \gamma}^{(1)}(\Delta)$.

Conversely, if w is a critical point of the restriction of E_ϕ to $\Omega_{p, \gamma}^{(1)}(\Delta)$, then the Gateaux derivative $dE_\phi(w)[V]$ vanishes for all $V \in T_w \Omega_{p, \gamma}^{(1)}(\Delta)$. Let's define:

$$(4.8) \quad \mathbf{T}_w = \left\{ V \in T_w \Omega_{p, \gamma}^{(1)} : V = \tau \cdot Y, \text{ for some } \tau \in H^1(I, \mathbb{R}) \text{ with } \tau(0) = \tau(1) = 0 \right\}.$$

Since Y is Killing in the metric g_R , an easy calculation shows that for all $w \in \Omega_{p, \gamma}^{(1)}$, the Gateaux derivative $dE_\phi(w)[V]$ vanishes for all $V \in \mathbf{T}_w$.

Moreover, for all $w \in \Omega_{p, \gamma}^{(1)}(\Delta)$ it is (see [9]):

$$T_w \Omega_{p, \gamma}^{(1)} = \mathbf{T}_w + T_w \Omega_{p, \gamma}^{(1)}(\Delta),$$

which implies $dE_\phi(w)[V] = 0$ for all $V \in T_w\Omega_{p,\gamma}^{(1)}$. This concludes the proof. \square

The functional E_ϕ of (4.7) is called the *energy* functional relative to the metric $\phi \cdot g_R$. The critical points of E_ϕ in $\Omega_{p,\gamma}^{(1)}$ (or equivalently in $\Omega_{p,\gamma}^{(1)}(\Delta)$, see [9]) will be called *horizontal geodesics* between p and γ with respect to the Riemannian metric $\phi \cdot g_R$.

In order to state properly our variational principle, we introduce an operator \mathcal{D} that *deforms* curves in $\Omega_{p,\gamma}^{(2)}$ into horizontal curves using the flow of Y .

Let \mathcal{D} be the map:

$$(4.9) \quad \mathcal{D} : \Omega_{p,\gamma}^{(1)} \longrightarrow \Omega_{p,\gamma}^{(1)}(\Delta)$$

defined by $\mathcal{D}(\sigma) = w$, where

$$(4.10) \quad w(t) = \psi(\sigma(t), \mathbf{r}_\sigma(t)),$$

and \mathbf{r}_σ is the unique solution on $[0, 1]$ of the Cauchy problem:

$$(4.11) \quad \mathbf{r}_\sigma' = -\frac{\langle \dot{\sigma}, Y \rangle}{\langle Y, Y \rangle}, \quad \mathbf{r}_\sigma(0) = 0.$$

Using the Killing property of Y it is easily checked that \mathcal{D} is well defined, i.e., the maximal solution of (4.11) is defined on the entire interval $[0, 1]$ and the corresponding curve w given by (4.10) is horizontal. Namely, using the fact that the differential $d_x\psi$ is an isometry, we compute easily:

$$(4.12) \quad \begin{aligned} \langle \dot{w}, Y(w) \rangle &= \langle d_x\psi(\sigma, \mathbf{r}_\sigma)[\dot{\sigma}], Y(\psi(\sigma, \mathbf{r}_\sigma)) \rangle + \mathbf{r}_\sigma' \langle Y(\psi(\sigma, \mathbf{r}_\sigma)), Y(\psi(\sigma, \mathbf{r}_\sigma)) \rangle = \\ &= \langle \dot{\sigma}, Y(\sigma) \rangle + \mathbf{r}_\sigma' \langle Y(\sigma), Y(\sigma) \rangle = 0. \end{aligned}$$

Observe that, if $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$, then (4.11) gives:

$$(4.13) \quad \mathbf{r}_\sigma' = \frac{k \mathcal{T}_\sigma}{\langle Y, Y \rangle}.$$

In Section 7 we will need to use the differential $d\mathcal{D}$ of \mathcal{D} on brachistochrones; the differentiability of \mathcal{D} and a formula for $d\mathcal{D}$ is established in the next:

Proposition 4.4. *The map \mathcal{D} is smooth around the regular points of $\mathcal{B}_{p,\gamma}^{(1)}(k)$. If σ is a curve of class C^1 in $\mathcal{B}_{p,\gamma}^{(1)}(k)$ and $\zeta \in T_\sigma\mathcal{B}_{p,\gamma}^{(1)}(k)$, the Gateaux derivative $d\mathcal{D}(\sigma)[\zeta]$ is given by:*

$$(4.14) \quad d\mathcal{D}(\sigma)[\zeta] = d_x\psi(\sigma, \mathbf{r}_\sigma) [\zeta + \tau_\zeta \cdot Y(\sigma)],$$

where $\tau_\zeta : [0, 1] \longrightarrow \mathbb{R}$ is the function:

$$(4.15) \quad \tau_\zeta(t) = - \int_0^t \frac{C_\zeta \langle Y, Y \rangle + 2k \mathcal{T}_\sigma \langle \nabla_\zeta Y, Y \rangle}{\langle Y, Y \rangle^2} dr,$$

where C_ζ is the constant $\langle \nabla_{\dot{\sigma}} \zeta, Y \rangle - \langle \zeta, \nabla_{\dot{\sigma}} Y \rangle$. In particular, if σ is a brachistochrone, then τ_ζ takes the following form:

$$(4.16) \quad \tau_\zeta(t) = -2k \mathcal{T}_\sigma \int_0^t \frac{\langle \nabla_\zeta Y, Y \rangle}{\langle Y, Y \rangle^2} dr.$$

Proof. The smooth dependence on σ of the solution \mathbf{r}_σ of (4.11) proves that \mathcal{D} is a smooth map. Formulas (4.14), (4.15) and (4.16) are easily obtained by differentiating (4.10) using (2.42), and keeping in mind that $d_x\psi(\sigma, \mathbf{r}_\sigma)[Y(\sigma)] = Y(\psi(\sigma, \mathbf{r}_\sigma))$. In particular, formula (4.16) follows immediately from (4.15) and Corollary 2.4. \square

Observe that formula (4.10) allows to extend the definition of the map \mathcal{D} to the space $\mathcal{B}_p^{(1)}(k)$ and with values in $\Omega_p^{(1)}$; these spaces were defined in (2.39) and (4.6). Obviously, Proposition 4.4 remains true for the extension.

Now everything is ready to state and prove the following:

Proposition 4.5 (First Variational Principle for Brachistochrones).

Let $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ be fixed. The following are equivalent:

1. σ is a brachistochrone of energy k between p and γ ;
2. σ is a local minimizer for the travel time;
3. $w = \mathcal{D}(\sigma) \in \Omega_{p,\gamma}^{(1)}(\Delta)$ is a horizontal geodesic between p and γ with respect to the Riemannian metric $\phi_k \cdot g_R$, where:

$$(4.17) \quad \phi_k = -\frac{\langle Y, Y \rangle}{k^2 + \langle Y, Y \rangle}.$$

Moreover, if one of the conditions above is satisfied, then $E_{\phi_k}(w) = \frac{1}{2}\mathcal{T}_\sigma^2$, where E_{ϕ_k} is the energy functional relative to the metric $\phi_k \cdot g_R$, given by:

$$(4.18) \quad E_{\phi_k}(w) = \frac{1}{2} \int_0^1 \phi_k(w) \langle \dot{w}, \dot{w} \rangle_{(R)} dt, \quad \forall w \in \Omega_{p,\gamma}^{(1)}.$$

Proof. The equivalence of conditions 1 and 2 follows from the fact that the brachistochrones of energy k between p and γ and the local minimizers for the travel time are characterized by the same differential equation (see Proposition 3.6 and Ref. [9, Definition 1.1, Corollary 3.2]).

The equivalence of condition 2 and 3 is based on the fact that, for $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ and $w = \mathcal{D}(\sigma)$, using (3.5), (4.10) and (4.11), one computes easily:

$$(4.19) \quad \begin{aligned} \phi_k(w) \langle \dot{w}, \dot{w} \rangle &= \\ &= -\frac{\langle Y(\sigma), Y(\sigma) \rangle}{k^2 + \langle Y(\sigma), Y(\sigma) \rangle} (\langle \dot{\sigma}, \dot{\sigma} \rangle + 2\langle \dot{\sigma}, Y(\sigma) \rangle \mathbf{r}_{\sigma'} + (\mathbf{r}_{\sigma'})^2 \langle Y(\sigma), Y(\sigma) \rangle) = \mathcal{T}_\sigma^2. \end{aligned}$$

Here we have used the facts that $\langle Y, Y \rangle$ is constant along the flow lines of Y , that $\psi(\cdot, t_0)$ is an isometry for all $t_0 \in \mathbb{R}$ and the conservation law of the energy of the Riemannian geodesics. Observe that, since Y is Killing in the metric $\phi_k \cdot g_R$, then a critical point of E_{ϕ_k} in $\Omega_{p,\gamma}^{(1)}(\Delta)$ is indeed a geodesic with respect to $\phi_k \cdot g_R$ (see [9]). It follows that the quantity $\phi_k(w) \langle \dot{w}, \dot{w} \rangle$ is constant along each horizontal geodesic w .

Recalling (2.43), integrating formula (4.19) yields:

$$(4.20) \quad F = -E_{\phi_k} \circ \mathcal{D}.$$

From (4.19) it follows that σ is a local minimizer for the travel time if and only if w is a local minimizer for the energy functional E_{ϕ_k} in $\Omega_{p,\gamma}^{(1)}(\Delta)$, i.e., if and only if w is a horizontal geodesic between p and γ with respect to $\phi_k \cdot g_R$.

The last statement of the thesis follows easily by integrating (4.19) over $[0, 1]$. \square

The result of Proposition 4.5 remains true for brachistochrones and horizontal geodesics with free endpoints in U_k . The correct statement of this fact is obtained by replacing the spaces $\mathcal{B}_{p,\gamma}^{(1)}(k)$ and $\Omega_{p,\gamma}^{(1)}(\Delta)$ respectively with $\mathcal{B}_p^{(1)}(k)$ and $\Omega_p^{(1)}(\Delta)$, which were defined in formulas (2.39) and (4.6).

5. THE SECOND VARIATION OF THE TRAVEL TIME

In this section we want to investigate the problem of whether a given stationary point σ in $\mathcal{B}_{p,\gamma}^{(1)}(k)$ for the travel time functional is a local minimum, maximum or a saddle point. To this aim, we need a second order variational formula for our variational problem.

In the first part of the Section we will discuss the abstract problem of relating the Hessians of smooth functions on Banach manifolds that are intertwined by a Banach manifold morphism; then we use the first part to determine the relation between the Hessian of the travel time T and the Hessian of the Riemannian action E_{ϕ_k} .

Let M be a Banach manifold and $f : M \rightarrow \mathbb{R}$ be a smooth map. If $x_0 \in M$ is a critical point for f , i.e., $df(x_0) = 0$, then it makes sense to define the *Hessian* of f at x_0 , denoted by $H^f(x_0)$, which is a continuous symmetric bilinear form on $T_{x_0}M$, in the following way.

Choose a coordinate system around x_0 , $\phi : U \subset M \mapsto U_0 \subset E$, where E is some Banach space. Define:

$$(5.1) \quad H^f(x_0)[v, w] = d^2(f \circ \phi^{-1})(\phi(x_0))[d\phi(x_0)[v], d\phi(x_0)[w]],$$

for $v, w \in T_{x_0}M$. Using the fact that x_0 is critical for f , it is easy to see that this definition will not depend on the chart (U, ϕ) . Indeed, it is easily seen that for every smooth curve $s \mapsto y_s \in M$ such that $y_0 = x_0$ and $y'_0 = v \in T_{x_0}M$, we have:

$$(5.2) \quad \frac{d^2(f(y_s))}{ds^2} \Big|_{s=0} = H^f(x_0)[v, v].$$

Formula (5.2) provides a simple way of computing $H^f(x_0)[v, v]$; the general formula for $H^f(x_0)[v, w]$ is easily obtained by polarization.

We now prove the following:

Lemma 5.1. *Let M and N be Banach manifolds and $\mathcal{D} : M \mapsto N$ be a smooth map; let $f : N \mapsto \mathbb{R}$ be a smooth function. If $x_0 \in M$ is such that $\mathcal{D}(x_0)$ a critical point for f , then x_0 is a critical point for $f \circ \mathcal{D}$, and the Hessians $H^f(\mathcal{D}(x_0))$ and $H^{f \circ \mathcal{D}}(x_0)$ are related by:*

$$(5.3) \quad H^f(\mathcal{D}(x_0))[d\mathcal{D}(x_0)[v], d\mathcal{D}(x_0)[w]] = H^{f \circ \mathcal{D}}(x_0)[v, w],$$

for all $v, w \in T_{x_0}M$.

Proof. Since both sides of (5.3) are symmetric, it suffices to prove the equality in the case $v = w$. Let $y(s)$, $s \in]-\varepsilon, \varepsilon[$ be a smooth curve in M such that $y(0) = x_0$ and $y'(0) = v$. Then, clearly, $\tilde{y} = \mathcal{D} \circ y$ is a smooth curve in N such that $\tilde{y}(0) = \mathcal{D}(x_0)$ and $\tilde{y}'(0) = d\mathcal{D}(x_0)[v]$. Using (5.2), we have:

$$H^f(\mathcal{D}(x_0))[d\mathcal{D}(x_0)[v], d\mathcal{D}(x_0)[v]] = \frac{d^2(f \circ \mathcal{D} \circ y)}{ds^2} \Big|_{s=0} = H^{f \circ \mathcal{D}}(x_0)[v, v],$$

which concludes the proof. \square

From Lemma 5.1 and formula (4.20), setting $f = E_{\phi_k}$, it follows immediately:

Corollary 5.2 (Second order variational principle for brachistochrones).

Let $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ be a brachistochrone and $w = \mathcal{D}(\sigma)$. Then, for all $\zeta_1, \zeta_2 \in T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$, we have:

$$(5.4) \quad H^F(\sigma)[\zeta_1, \zeta_2] = -H^{E_{\phi_k}}(w)[d\mathcal{D}(w)[\zeta_1], d\mathcal{D}(w)[\zeta_2]]. \quad \square$$

From (2.43) and (5.2) we obtain easily:

$$(5.5) \quad H^F(\sigma) = -\mathcal{T}_\sigma \cdot H^T(\sigma)$$

for all brachistochrone $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$.

From (5.4) and (5.5) we obtain also:

$$(5.6) \quad H^T(\sigma)[\zeta_1, \zeta_2] = \mathcal{T}_\sigma^{-1} \cdot H^{E_{\phi_k}}(w)[d\mathcal{D}(w)[\zeta_1], d\mathcal{D}(w)[\zeta_2]],$$

for all brachistochrone $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ and all $\zeta_1, \zeta_2 \in T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$.

6. THE RIEMANNIAN MORSE INDEX THEOREM

For Riemannian geodesics, the classical Morse Index Theorem (see References [1, 3, 12, 18] for the different versions of this Theorem) relates the index of the action functional with some geometrical properties of the geodesic. The main ingredients for the theory are given by the curvature tensor of the metric and the concepts of *Jacobi fields* and *conjugate* or *focal* points along a geodesic.

In view to applications to the brachistochrone problem, in this section we quickly review some known results about the Morse Index Theorem for Riemannian geodesics joining a curve with a point, as presented, for instance, in [12]. Then, we prove a different version of this theorem in the case of an orthogonal geodesic between the integral line of a Killing vector field and a point.

In order to simplify the formulas, in this section we interchange the role of p and γ , that is, we consider curves starting at the curve γ and ending at the point p . Clearly, the final results (Theorems 6.8 and 6.9) will not be affected by this change. Moreover, all the results and the formulas of the previous sections remain true after changing the variable t with $1 - t$ in the interval $[0, 1]$, and, in particular, the role of the endpoints $t = 0$ and $t = 1$ will be interchanged. To avoid confusion, in this section we will use the symbols $\Omega_{\gamma,p}^{(1)}$ and $\Omega_{\gamma,p}^{(1)}(\Delta)$ to indicate the spaces of curves in U_k of class H^1 from γ to p . If we denote by \mathcal{O} the *direction reversing map* for curves $w : [0, 1] \mapsto \mathcal{M}$, i.e.,

$$(6.1) \quad \mathcal{O}(w)(t) = w(1 - t),$$

then clearly $\Omega_{\gamma,p}^{(1)} = \mathcal{O}(\Omega_{p,\gamma}^{(1)})$ and $\Omega_{\gamma,p}^{(1)}(\Delta) = \mathcal{O}(\Omega_{p,\gamma}^{(1)}(\Delta))$. Observe that, for all $i \in \mathbb{N}$, the restriction of \mathcal{O} to the Sobolev manifold $H^i([0, 1], \mathcal{M})$ is smooth, and its differential is formally given by:

$$d\mathcal{O}[V](t) = V(1 - t), \quad V \in H^i([0, 1], T\mathcal{M}).$$

Observe also that the energy functional E_{ϕ_k} can be defined in $\Omega_{\gamma,p}^{(1)}$ by the same formula (4.18); obviously, a curve w is a critical point for E_{ϕ_k} in $\Omega_{\gamma,p}^{(1)}$ if and only if $\mathcal{O}(w)$ is a critical point for E_{ϕ_k} in $\Omega_{p,\gamma}^{(1)}$. In this case, we have:

$$(6.2) \quad H^{E_{\phi_k}}(w)[V, W] = H^{E_{\phi_k}}(\mathcal{O}(w))[d\mathcal{O}[V], d\mathcal{O}[W]], \quad \forall V, W \in T_w\Omega_{\gamma,p}^{(1)}.$$

By Lemma 4.3, we know that the critical points of the Riemannian energy functional E_{ϕ_k} corresponding to the metric $\phi_k \cdot g_R$ on the spaces $\Omega_{\gamma,p}^{(1)}$ and $\Omega_{\gamma,p}^{(1)}(\Delta)$ are the same. However, given a horizontal geodesic w between p and γ , the Morse index of E_{ϕ_k} at w (see Definition 6.5) in the Hilbert manifold $\Omega_{\gamma,p}^{(1)}(\Delta)$ may be strictly less than the Morse index of E_{ϕ_k} at w in the manifold $\Omega_{\gamma,p}^{(1)}$. The purpose of this section is to prove that the two indices are indeed equal; we accomplish this result by proving an index theorem for the Morse index $m(w, E_{\phi_k})$ restricted to the space $T_w\Omega_{\gamma,p}^{(1)}(\Delta)^\perp$, defined by:

$$(6.3) \quad T_w\Omega_{\gamma,p}^{(1)}(\Delta)^\perp = \left\{ V \in T_w\Omega_{\gamma,p}^{(1)}(\Delta) \mid \phi_k(w) \cdot \langle V, \dot{w} \rangle_{(R)} \equiv C_V \text{ (const.)} \right\}.$$

Observe that $T_w\Omega_{\gamma,p}^{(1)}(\Delta)^\perp$ is a (closed) Hilbert subspace of $T_w\Omega_{\gamma,p}^{(1)}(\Delta)$; moreover, if w is a horizontal geodesic with respect to the metric $\phi_k \cdot g_R$ in $\Omega_{\gamma,p}^{(1)}$, then, for a vector field $V \in T_w\Omega_{\gamma,p}^{(1)}(\Delta)$ we have:

$$(6.4) \quad V \in T_w\Omega_{\gamma,p}^{(1)}(\Delta)^\perp \iff \langle \nabla_{\dot{w}}^{\{k\}} V, \dot{w} \rangle_{(R)} = 0,$$

where $\nabla^{\{k\}}$ is the covariant derivative of the Levi-Civita connection of the Riemannian metric $\phi_k \cdot g_R$.

Indeed, if $V \in T_w\Omega_{\gamma,p}^{(1)}(\Delta)^\perp$, then, since $\nabla_{\dot{w}}^{\{k\}} \dot{w} = 0$, it is

$$0 = \frac{d}{dt} [\phi_k(w) \cdot \langle V, \dot{w} \rangle_{(R)}] = \phi_k(w) \cdot \langle \nabla_{\dot{w}}^{\{k\}} V, \dot{w} \rangle_{(R)}.$$

On the other hand, if $0 = \phi_k(w) \cdot \langle \nabla_{\dot{w}}^{\{k\}} V, \dot{w} \rangle_{(R)} = \frac{d}{dt} [\phi_k(w) \cdot \langle V, \dot{w} \rangle_{(R)}]$, the quantity $\phi_k(w) \cdot \langle V, \dot{w} \rangle_{(R)}$ is constant and (6.4) is proven.

In particular, since $V(1) = 0$ (recall that we are considering curves ending at the fixed point p), if w is a horizontal geodesic and $V \in T_w\Omega_{\gamma,p}^{(1)}(\Delta)^\perp$, then $C_V = 0$. Hence, a vector field $V \in T_w\Omega_{\gamma,p}^{(1)}(\Delta)$ belongs to $T_w\Omega_{\gamma,p}^{(1)}(\Delta)^\perp$ if and only if it is everywhere perpendicular to w , which is the reason for the notation.

Remark 6.1. From (6.4) it is easy to see that, if we think of the elements in $T_w\Omega_{\gamma,p}^{(1)}$ as variational vector fields relative to variations w_s of the horizontal geodesic w , then the condition

$V \in T_w \Omega_{\gamma,p}^{(1)}(\Delta)^\perp$ means that, up to infinitesimals of order larger than 1, the curves w_s are horizontal, and they are parameterized by a constant multiple of arclength:

$$\frac{d}{ds} \Big|_{s=0} \left[\phi_k(w_s) \langle \dot{w}_s, Y \rangle_{(R)} \right] = \phi_w(w) \left(\langle \nabla_{\dot{w}}^{\{k\}} V, Y \rangle_{(R)} - \langle V, \nabla_{\dot{w}}^{\{k\}} Y \rangle_{(R)} \right) = 0,$$

$$\frac{d}{ds} \Big|_{s=0} \left[\phi_k(w_s) \langle \dot{w}_s, \dot{w}_s \rangle_{(R)} \right] = 2\phi_k(w) \cdot \langle \nabla_{\dot{w}}^{\{k\}} V, \dot{w} \rangle_{(R)} = 0.$$

We can easily write (6.4) in terms of the Lorentzian structure, by differentiating the above expression using the Lorentzian covariant derivative. Given a horizontal geodesic w and $V \in T_w \Omega_{\gamma,p}^{(1)}(\Delta)$, we have that $V \in T_w \Omega_{\gamma,p}^{(1)}(\Delta)^\perp$ if and only if the following equation holds:

$$(6.5) \quad \langle \nabla \phi_k(w), V \rangle \cdot \langle \dot{w}, \dot{w} \rangle + 2\phi_k(w) \cdot \langle \nabla_{\dot{w}} V, \dot{w} \rangle = 0.$$

We recall the basic facts concerning the Morse Index Theorem for Riemannian geodesics between a point and a curve, as it is presented, for instance, in Ref. [12].

Given a horizontal geodesic w in between p and γ with respect to the Riemannian metric $\phi_k \cdot g_R$, let $\nabla^{\{k\}}$ and $R^{\{k\}}$ denote respectively the Levi-Civita connection and the curvature tensor (chosen with the same sign convention as in (2.1)) of the metric $\phi_k \cdot g_R$, and let $\mathcal{J}_w^{\{k\}}$ be the finite dimensional vector space of all the Jacobi fields J along w with respect to $\phi_k \cdot g_R$, i.e., all smooth vector fields satisfying the second order differential equation:

$$(6.6) \quad \nabla_{\dot{w}}^{\{k\}} \nabla_{\dot{w}}^{\{k\}} J - R^{\{k\}}(\dot{w}, J) \dot{w} = 0.$$

We recall that, in analogy with the Riemannian case, given a submanifold Σ of \mathcal{M} whose tangent bundle $T\Sigma$ is non degenerate, i.e., the restriction of g to the tangent space $T_q \Sigma$ is non degenerate for all $q \in \Sigma$, one can define the *second fundamental form* S^Σ (also known as the *shape tensor* of Σ) as follows. For each $q \in \Sigma$ and each vector $n \in T_q \Sigma^\perp$, the second fundamental form of Σ in the direction of n is the bilinear form $S_n^\Sigma : T_q \Sigma \times T_q \Sigma \longrightarrow \mathbb{R}$ defined by:

$$(6.7) \quad S_n^\Sigma(v_1, v_2) = \langle n, \nabla_{v_1} V_2 \rangle,$$

where V_2 is any smooth vector field on Σ that takes value v_2 at q . One can show that S_n^Σ is well defined (i.e., formula (6.7) does not indeed depend on the choice of the extension V_2 of v_2), and it is *symmetric* (see for instance [2] and [19]).

In the following, we will denote by S^γ the second fundamental form of the timelike submanifold $\gamma(\mathbb{R})$ of \mathcal{M} .

Let $\mathcal{J}_w^{\{k\}}(\gamma)$ denote the subspace of $\mathcal{J}_w^{\{k\}}$ consisting of all γ -Jacobi fields i.e., all the Jacobi fields J along w satisfying:

1. $J(0) \parallel Y(w(0))$;
2. $\langle \nabla_{\dot{w}(0)} J, Y \rangle + S_{\dot{w}(0)}^\gamma(J(0), Y) = \langle \nabla_{\dot{w}(0)} J, Y \rangle + \langle \dot{w}(0), \nabla_{J(0)} Y \rangle = 0$.

Finally, for $t_0 \in]0, 1]$, we denote by $\mathcal{J}_w^{\{k\}}(\gamma, t_0)$ the set of γ -Jacobi fields J along w that vanish at t_0 :

3. $J(t_0) = 0$.

A point $w(t_0)$ along w is said to be a γ -focal point if $\dim(\mathcal{J}_w^{\{k\}}(t_0)) > 0$; the *multiplicity* of the a γ -focal point $w(t_0)$ is the dimension of $\mathcal{J}_w^{\{k\}}(t_0)$ (which is clearly finite, because the Jacobi fields are solutions of a second order linear system of differential equations).

Remark 6.2. It is well known that the set of γ -focal points along every Riemannian geodesic is discrete, hence there is only a finite number of γ -focal points along each compact portion of a geodesic. For the reader's convenience, we sketch a simple proof of this fact based on [19, Ex. 8, p. 299]. The set of γ -Jacobi field along a given geodesic w has dimension equal to $m = \dim(M)$. If J_1, J_2, \dots, J_m is a family of linearly independent γ -Jacobi fields and E_1, E_2, \dots, E_m is a parallelly transported orthonormal basis along w , then one considers the smooth function

$g(t) = \det(\langle J_i(t), E_j(t) \rangle)$. Using elementary arguments, one proves that t_0 is a zero of order d for g , i.e., $g(t_0) = g'(t_0) = \dots g^{(d-1)}(t_0) = 0$ and $g^{(d)}(t_0) \neq 0$, if and only if $w(t_0)$ is a γ -focal point of multiplicity d . In particular, the set of γ -focal points is discrete, as is the set of simple zeroes of a smooth function.

Equation (6.6) is obtained by linearizing the geodesic equation in the metric $\phi_k \cdot g_R$; hence, it is satisfied by vector fields along w that correspond to variations w_s , $s \in]-\varepsilon, \varepsilon[$ for some $\varepsilon > 0$, of w consisting of geodesics. Loosely speaking, the arrow-head of J traces out infinitesimally close neighboring geodesics to w .

The condition 1 means that, in a first order approximation, these geodesics start on γ ; condition 3 means that they pass through $w(t_0)$. Condition 2 means that these geodesics start orthogonally at γ ; observe that orthogonality to the vector field Y is equivalent in the three metrics g , g_R and $\phi_k \cdot g_R$, and for this reason it is possible to write this condition using the Lorentzian Levi-Civita connection ∇ and the Lorentzian second fundamental form S^γ of γ . Using the Riemannian metric $\phi_k \cdot g_R$, condition 2 can also be written as:

$$2b. \langle \nabla_{\dot{w}(0)}^{\{k\}} J, Y \rangle_{(R)} + \langle \dot{w}(0), \nabla_{J(0)}^{\{k\}} Y \rangle_{(R)} = 0.$$

Remark 6.3. Observe that, since Y is Killing, we obtain easily that, if J satisfies the differential equation 6.6, then the condition $\langle \nabla_{\dot{w}} J, Y \rangle + \langle \dot{w}, \nabla_J Y \rangle = 0$ is satisfied identically on $[0, 1]$ provided that it is satisfied at one single point $t_0 \in [0, 1]$. Indeed, using the fact that Killing vector fields satisfy the Jacobi equation (see [19, Lemma 26, p. 252]), it is easy to see that the quantity $\langle \nabla_{\dot{w}} J, Y \rangle + \langle \dot{w}, \nabla_J Y \rangle = \langle \nabla_{\dot{w}} J, Y \rangle - \langle J, \nabla_{\dot{w}} Y \rangle$ is constant:

$$(6.8) \quad \begin{aligned} \frac{d}{dt} (\langle \nabla_{\dot{w}} J, Y \rangle - \langle J, \nabla_{\dot{w}} Y \rangle) &= \langle \nabla_{\dot{w}}^2 J, Y \rangle - \langle J, \nabla_{\dot{w}}^2 Y \rangle = \\ &= \langle R(\dot{w}, J) \dot{w}, Y \rangle - \langle J, R(\dot{w}, Y) \dot{w} \rangle = 0, \end{aligned}$$

where the last equality follows easily from well known symmetry properties of the curvature tensor R .

From Remark 6.3 and formula (4.5), we obtain immediately the following characterization of the γ -Jacobi fields along a horizontal geodesic w :

Lemma 6.4. *Let w be a horizontal geodesic in $\Omega_{\gamma,p}^{(1)}(\Delta)$ and W a Jacobi field along w . Then, W is a γ -Jacobi field if and only if $W \in T_w \Omega_{\gamma,p}^{(1)}(\Delta)$. \square*

Given a horizontal geodesic w , we denote by $I^{\{k\}}$ the *index form* on $T_w \Omega_{\gamma,p}^{(1)}$, or more in general on $T_w H^1([0, 1], \mathcal{M})$, given by the symmetric bilinear form:

$$(6.9) \quad I^{\{k\}}(V_1, V_2) = \int_0^1 \phi_k(w) \left(\langle \nabla_{\dot{w}}^{\{k\}} V_1, \nabla_{\dot{w}}^{\{k\}} V_2 \rangle_{(R)} + \langle R^{\{k\}}(\dot{w}, V_1) \dot{w}, V_2 \rangle_{(R)} \right) dt.$$

The symmetry of $I^{\{k\}}$ follows easily from the symmetry properties of the curvature tensor $R^{\{k\}}$; moreover, from the fundamental Lemma of Calculus of Variations, a simple integration by parts in (6.9) shows that a vector field W along w is a Jacobi field if and only if

$$(6.10) \quad I^{\{k\}}(W, V) = 0$$

for all smooth vector field V along w such that $V(0) = V(1) = 0$.

We recall the definition of the Morse index at a critical point of a C^2 -functional on a Hilbert manifold:

Definition 6.5. Let M be a Hilbert manifold, $f : M \rightarrow \mathbb{R}$ be a map of class C^2 , x_0 a critical point for f in M and X a Hilbert subspace of $T_{x_0} M$. The Morse index $m(x_0, f, X)$ of f at x_0 in X is the dimension of a maximal subspace of X on which the Hessian $H^f(x_0)$ is *negative* definite. Whenever there is no danger of confusion, we will denote by $m(x_0, f) = m(x_0, f, T_{x_0} M)$ the Morse index of f at x_0 in the entire tangent space $T_{x_0} M$.

The *kernel* of $H^f(x_0)$, denoted by $\text{Ker}(H^f(x_0))$ is the Hilbert subspace of $T_{x_0}M$ consisting of vectors X such that $H^f(x_0)[X, Y] = 0$ for all $Y \in T_{x_0}M$.

Roughly speaking, the Morse index $m(x_0, f)$ gives the number of *essentially different* directions in which the value of the functional f increases from the value $f(x_0)$. Clearly, if $m(x_0, f) = 0$, then x_0 is a local maximum for f .

Remark 6.6. Observe that, for all subspace $X \subset T_{x_0}M$, we have

$$(6.11) \quad m(x_0, f, X) \leq m(x_0, f).$$

On the other hand, suppose that X is a closed subspace of $T_{x_0}M$ and that the restriction of $H^f(x_0)$ to X is nondegenerate. Let X_1 be the orthogonal space to X relatively to the bilinear form $H^f(x_0)$, which is the closed subspace of $T_{x_0}M$ defined by:

$$X_1 = \left\{ V_1 \in T_{x_0}M : H^f(x_0)[V, V_1] = 0 \ \forall V \in X \right\}.$$

If the restriction of $H^f(x_0)$ to X_1 is positive semidefinite, then $m(x_0, f, X) = m(x_0, f)$.

If w is a horizontal geodesic between p and γ with respect to the Riemannian metric $\phi_k \cdot g_R$, or equivalently, w is a critical point for E_{ϕ_k} in $\Omega_{\gamma, p}^{(1)}$, then the Hessian $H^{E_{\phi_k}}(w)$ is computed easily in terms of the metric $\phi_k \cdot g_R$ as:

$$(6.12) \quad H^{E_{\phi_k}}(w)[V, V] = I^{\{k\}}(V, V) - \phi_k(w(0)) \langle \nabla_{V(0)}^{\{k\}} V, \dot{w}(0) \rangle_{(R)}.$$

Since $V(0)$ is tangent to the curve γ and $\dot{w}(0)$ is orthogonal to γ , then the term

$$\phi_k(w(0)) \langle \nabla_{V(0)}^{\{k\}} V, \dot{w}(0) \rangle_{(R)}$$

is tensorial in V , i.e., it only depends on the value $V(0)$. This is precisely the second fundamental form of the curve γ in the direction of the normal vector $\dot{w}(0)$ with respect to the metric $\phi_k \cdot g_R$.

We can give a different expression of the Hessian $H^{E_{\phi_k}}(w)$ in terms of the Lorentzian metric g . This is done by direct computation in the following:

Proposition 6.7. *Let $w \in \Omega_{p, \gamma}^{(1)}(\Delta)$ be a horizontal geodesic between p and γ with respect to the Riemannian metric $\phi_k \cdot g_R$. Then, the Hessian $H^{E_{\phi_k}}(w)$ is given by the following symmetric bilinear map on $T_w \Omega_{p, \gamma}^{(1)}(\Delta)$:*

$$(6.13) \quad \begin{aligned} H^{E_{\phi_k}}(w)[V, V] &= \int_0^1 \phi_k(w) \left[\langle \nabla_{\dot{w}} V, \nabla_{\dot{w}} V \rangle + \langle R(V, \dot{w}) V, \dot{w} \rangle \right] dt + \\ &+ \int_0^1 \left[2 \langle \nabla \phi_k(w), V \rangle \langle \nabla_{\dot{w}} V, \dot{w} \rangle + \frac{1}{2} \langle H^{\phi_k}(w) V, V \rangle \langle \dot{w}, \dot{w} \rangle \right] dt + \\ &+ \phi(w(1)) \cdot S_{\dot{w}(1)}^\gamma(V(1), V(1)). \end{aligned}$$

Proof. The geodesic equation for the metric $\phi_k \cdot g_R$ is easily computed as the Euler–Lagrange equation for the functional E_{ϕ_k} , and it is given by:

$$(6.14) \quad \nabla_{\dot{w}} [\phi_k(w) \dot{w}] = \frac{1}{2} \nabla \phi_k(w) \langle \dot{w}, \dot{w} \rangle.$$

In analogy with the proof of Proposition A.3, let V be a fixed vector field in $T_w \Omega_{p, \gamma}^{(1)}(\Delta)$ and let w_s denote a variation of w in $\Omega_{p, \gamma}^{(1)}(\Delta)$ such that $V = \frac{d}{ds} \big|_{s=0} w_s$.

Then, we compute as follows:

$$\begin{aligned}
 H^{E_{\phi_k}}(w)[V, V] &= \frac{d^2}{ds^2} \Big|_{s=0} E_{\phi_k}(w_s) = \\
 (6.15) \quad &= \int_0^1 \left(\frac{1}{2} \frac{d^2}{ds^2} \Big|_{s=0} [\phi_k(w_s)] \langle \dot{w}, \dot{w} \rangle + 2 \langle \nabla \phi_k(w), V \rangle \langle \nabla_{\dot{w}} V, \dot{w} \rangle \right) dt + \\
 &+ \int_0^1 \left(\phi_k(w) \left\langle \frac{D}{ds} \frac{D}{dt} \frac{d}{ds} w_s, \dot{w} \right\rangle + \phi_k(w) \langle \nabla_{\dot{w}} V, \nabla_{\dot{w}} V \rangle \right) dt.
 \end{aligned}$$

Using (6.14) and the commutation relations (A.5), we have:

$$\begin{aligned}
 \int_0^1 \phi_k(w) \left\langle \frac{D}{ds} \frac{D}{dt} \frac{d}{ds} w_s, \dot{w} \right\rangle dt &= \int_0^1 \phi_k(w) \langle R(V, \dot{w}) V, \dot{w} \rangle dt + \\
 (6.16) \quad &- \frac{1}{2} \int_0^1 \left\langle \frac{D}{ds} \frac{d}{ds} w_s, \nabla \phi_k(w) \right\rangle \langle \dot{w}, \dot{w} \rangle dt + \phi_k(w) \left\langle \frac{D}{ds} \frac{d}{ds} w_s, \dot{w} \right\rangle \Big|_0^1.
 \end{aligned}$$

Keeping in mind that $w_s(0) \equiv p$ and arguing as in the proof of Proposition A.3 (see formula A.15), the boundary term in (6.16) can be computed as:

$$\begin{aligned}
 \phi_k(w) \left\langle \frac{D}{ds} \frac{d}{ds} w_s, \dot{w} \right\rangle \Big|_0^1 &= \phi_k(w(1)) \frac{\langle V(1), Y(w(1)) \rangle}{\langle Y(w(1)), Y(w(1)) \rangle} \langle \nabla_{V(1)} Y, \dot{w}(1) \rangle = \\
 (6.17) \quad &= \phi_k(w(1)) S_{\dot{w}(1)}^\gamma(V(1), V(1)).
 \end{aligned}$$

Finally, we have:

$$(6.18) \quad \int_0^1 \frac{d^2}{ds^2} \Big|_{s=0} [\phi_k(w_s)] \langle \dot{w}, \dot{w} \rangle dt = \int_0^1 \left[\langle H^{\phi_k}(w) V, V \rangle + \langle \nabla \phi_k(w), \frac{D}{ds} \frac{d}{ds} w_s \rangle \right] dt.$$

Formula (6.13) follows from (6.15), (6.16), (6.17) and (6.18). \square

Let's now go back to the study of the second variation of E_{ϕ_k} in terms of the Riemannian metric $\phi_k \cdot g_R$. Using integration by parts in the Index formula (6.9), it is easy to see that the set of γ -Jacobi fields $\mathcal{J}_w^{\{k\}}(t_0)$ can be also described as the kernel of the Hessian $H^{E_{\phi_k}}(w)$ restricted to the interval $[0, t_0]$; in particular:

$$(6.19) \quad \mathcal{J}_w^{\{k\}} = \text{Ker} (H^{E_{\phi_k}}(w)).$$

The *geometric index* $\mu^{\{k\}}(w)$ of the horizontal geodesic w is defined as the natural number:

$$(6.20) \quad \mu^{\{k\}}(w) = \sum_{t_0 \in [0, 1]} \dim \left(\mathcal{J}_w^{\{k\}}(t_0) \right).$$

Recall from Remark 6.2 that the number of γ -focal points along w is finite, hence the sum in (6.20) is finite.

The Morse Index Theorem says that, if p is not a γ -focal point along w , the Morse index $m(w, E_{\phi_k})$ of E_{ϕ_k} in the space $T_w \Omega_{\gamma, p}^{(1)}$ is given by the number of γ -focal points along w , counted with multiplicity:

Theorem 6.8. *Let w be a critical point of E_{ϕ_k} in $\Omega_{\gamma, p}^{(1)}$, i.e., a geodesic from γ to p in the metric $\phi_k \cdot g_R$ that starts orthogonally to γ . Then, the Morse index $m(w, E_{\phi_k})$ is finite; moreover, if p is not a γ -focal point along w , we have:*

$$(6.21) \quad m(w, E_{\phi_k}) = \mu^{\{k\}}(w). \quad \square$$

Theorem 6.8 is obtained as a special case of [12, The Index Theorem, p. 342]. Observe that Theorem 6.8 holds without any assumption that γ be the integral line of a Killing vector field.

In the rest of this section we will prove that, given a horizontal geodesic w in $\Omega_{\gamma,p}^{(1)}$, then $m(w, E_{\phi_k})$ is equal to the Morse index $\bar{m}(w, E_{\phi_k})$ of the restriction of the Hessian $H^{E_{\phi_k}}$ on the space $T_w\Omega_{\gamma,p}^{(1)}(\Delta)^\perp$. Observe that, by (6.11), we have

$$\bar{m}(w, E_{\phi_k}) = m(w, E_{\phi_k}, T_w\Omega_{\gamma,p}^{(1)}(\Delta)^\perp) \leq m(w, E_{\phi_k}).$$

The desired result will follow immediately from our next theorem, that we state in a general form for future reference:

Theorem 6.9 (Second Morse Index Theorem for Horizontal Geodesics).

Let (\mathcal{M}, \tilde{g}) be a complete Riemannian manifold, Y a never vanishing complete Killing vector field on \mathcal{M} , $\gamma : \mathbb{R} \rightarrow \mathcal{M}$ an integral curve of Y , and $p \in \mathcal{M}$ be a point in $\mathcal{M} \setminus \gamma(\mathbb{R})$.

Let $\tilde{\Delta} = Y^\perp$ be the orthogonal distribution to Y ; moreover let $\Omega_{\gamma,p}^{(1)}$, $\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})$ denote the spaces:

$$\begin{aligned} \Omega_{\gamma,p}^{(1)} &= \left\{ w \in H^1([0, 1], \mathcal{M}) \mid w(0) \in \gamma(\mathbb{R}), w(1) = p \right\}, \\ \Omega_{\gamma,p}^{(1)}(\tilde{\Delta}) &= \left\{ w \in \Omega_{\gamma,p}^{(1)} \mid \tilde{g}(\dot{w}, Y) \equiv 0 \right\}; \end{aligned}$$

and, for $w \in \Omega_{\gamma,p}^{(1)}(\tilde{\Delta})$, let $T_w\Omega_{\gamma,p}^{(1)}$, $T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})$ and $T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\perp$ be defined in the obvious way (see formulas (2.11), (4.5) and (6.3)).

Let \tilde{E} denote the energy functional of the metric \tilde{g} in the space $\Omega_{\gamma,p}^{(1)}$; let w be a critical point of \tilde{E} in $\Omega_{\gamma,p}^{(1)}$ (or, equivalently, in $\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})$), and let $H^{\tilde{E}}(w)$ be the Hessian of \tilde{E} at w .

Then, if p is not a γ -focal point along w , the three indices are equal:

$$(6.22) \quad m(w, H^{\tilde{E}}) = m(w, H^{\tilde{E}}, T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})) = m(w, H^{\tilde{E}}, T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\perp).$$

Proof. The condition that the Killing vector field Y is never vanishing is needed to prove that the space $\Omega_{p,\gamma}^{(1)}(\tilde{\Delta})$ is a smooth submanifold of $\Omega_{p,\gamma}^{(1)}$ (see for instance Ref. [9]).

We start proving the second equality in (6.22); we denote by $\tilde{\nabla}$ and \tilde{R} respectively the covariant derivative and the curvature tensor of the Levi-Civita connection of \tilde{g} ; moreover, let \tilde{I} denote the index form in $T_w\Omega_{\gamma,p}^{(1)}$ with respect to the metric \tilde{g} , defined as in (6.9). Moreover, let $\tilde{\mu}(w)$ be the geometric index of the geodesic w in the metric \tilde{g} , defined as in (6.20).

Let $T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\parallel$ be defined by:

$$T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\parallel = \left\{ V \in T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta}) : V = \lambda \cdot \dot{w} \text{ for some } \lambda \in H^1([0, 1], \mathbb{R}) \right\}.$$

Clearly, $T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta}) = T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\perp \oplus T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\parallel$. Observe that, since $\dot{w}(0)$ is orthogonal to γ , then $V^\parallel(0) = 0$ for all $V^\parallel \in T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\parallel$.

Let $V^\perp \in T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\perp$ and $V^\parallel \in T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\parallel$ be fixed; using the fact that $V^\parallel(1) = 0$, $\tilde{g}(\tilde{R}(\dot{w}, \cdot)\dot{w}, \dot{w}) = 0$ and that $\tilde{g}(\tilde{\nabla}_{\dot{w}}V^\perp, \dot{w}) = \frac{d}{dt}\tilde{g}(V^\perp, \dot{w}) = 0$, it is easy to see that $H^{\tilde{E}}(w)[V^\perp, V^\parallel] = 0$.

This implies that $T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\perp$ and $T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\parallel$ are orthogonal with respect to the bilinear form $H^{\tilde{E}}(w)$; in particular, it is:

$$m(w, H^{\tilde{E}}, T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})) = m(w, H^{\tilde{E}}, T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\perp) + m(w, H^{\tilde{E}}, T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\parallel).$$

It is easy to see that $m(w, H^{\tilde{E}}, T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\parallel) = 0$; indeed, for $V^\parallel = \lambda \cdot \dot{w}$, since $V^\parallel(0) = 0$, from (6.12) we get:

$$H^{\tilde{E}}(w)[V, V] = \tilde{I}(V, V) = \int_0^1 \lambda'(t)^2 \cdot \tilde{g}(\dot{w}(t), \dot{w}(t)) dt \geq 0,$$

and since $\tilde{g}(\dot{w}, \dot{w}) > 0$ and $\lambda(0) = \lambda(1) = 0$, the above inequality implies that $H^{\tilde{E}}(w)$ is positive definite in $T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\parallel$, and so $m(w, H^{\tilde{E}}, T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})^\parallel) = 0$.

This proves the second equality in (6.22).

To prove the first equality, we prove that

$$(6.23) \quad m(w, H^{\tilde{E}}, T_w \Omega_{\gamma, p}^{(1)}(\tilde{\Delta})) = \tilde{\mu}(w),$$

and the conclusion will follow directly from Theorem 6.8.

To this goal, we will use also some abstract arguments in functional analysis on Hilbert spaces, and we introduce the following notation.

For all $t \in]0, 1]$, let $(\mathbf{T}_t, \langle \cdot, \cdot \rangle_t)$ be a real Hilbert space with relative inner product, defined by:

$$(6.24) \quad \mathbf{T}_t = \left\{ \zeta \in H^1([0, t], T\mathcal{M}) \text{ vector field along } w|_{[0, t]} : \right. \\ \left. \zeta(0) \parallel Y(w(0)), \zeta(t) = 0, \tilde{g}(\tilde{\nabla}_{\dot{w}} \zeta, Y) - \tilde{g}(\zeta, \tilde{\nabla}_{\dot{w}} Y) \equiv 0 \right\};$$

$$(6.25) \quad \langle \zeta_1, \zeta_2 \rangle_t = \int_0^t \tilde{g}(\tilde{\nabla}_{\dot{w}} \zeta_1, \tilde{\nabla}_{\dot{w}} \zeta_2) \, dr.$$

Observe that $\langle \cdot, \cdot \rangle_t$ is non degenerate on \mathbf{T}_t , because of the condition $\zeta(t) = 0$. Let $\| \cdot \|_t = \langle \cdot, \cdot \rangle_t^{\frac{1}{2}}$ be the relative norm.

We also define a continuous symmetric bilinear form \mathbf{H}_t on \mathbf{T}_t , by:

$$(6.26) \quad \mathbf{H}_t(\zeta_1, \zeta_2) = \int_0^t [\tilde{g}(\tilde{\nabla}_{\dot{w}} \zeta_1, \tilde{\nabla}_{\dot{w}} \zeta_2) + \tilde{g}(\tilde{R}(\dot{w}, \zeta_1) \dot{w}, \zeta_2)] \, dr - \tilde{g}(\dot{w}(0), \tilde{\nabla}_{\zeta_1(0)} \zeta_2);$$

observe that for $t = 1$, the Hilbert space $(\mathbf{T}_t, \langle \cdot, \cdot \rangle_t)$ coincide with $T_w \Omega_{\gamma, p}^{(1)}(\Delta)$ and the bilinear form \mathbf{H}_t is precisely the Hessian $H^{\tilde{E}}(w)$. The symmetry of \mathbf{H}_t is easily obtained using the symmetry of the curvature tensor and of the second fundamental form of γ . Observe also that, since $\zeta_1(0)$ and $\zeta_2(0)$ are multiples of the Killing field Y , then we have:

$$(6.27) \quad \tilde{g}(\dot{w}(0), \tilde{\nabla}_{\zeta_1(0)} \zeta_2) = - \frac{\tilde{g}(\zeta_1(0), Y(w(0))) \cdot \tilde{g}(\zeta_2(0), Y(w(0)))}{\tilde{g}(Y(w(0)), Y(w(0)))^2} \tilde{g}(Y(w(0)), \tilde{\nabla}_{\dot{w}(0)} Y).$$

Using the Riesz representation theorem, we can write \mathbf{H}_t as:

$$(6.28) \quad \mathbf{H}_t(\zeta_1, \zeta_2) = \langle \mathbf{L}_t[\zeta_1], \zeta_2 \rangle_t,$$

where \mathbf{L}_t is a self-adjoint linear operator on \mathbf{T}_t .

Comparing (6.25) and (6.26), we see that we can write:

$$(6.29) \quad \mathbf{H}_t = \mathbf{I}_t - \mathbf{K}_t,$$

where \mathbf{I}_t is the identity on \mathbf{T}_t and \mathbf{K}_t is the self-adjoint operator on \mathbf{T}_t defined by:

$$(6.30) \quad \langle \mathbf{K}_t[\zeta_1], \zeta_2 \rangle_t = - \int_0^t \tilde{g}(\tilde{R}(\dot{w}, \zeta_1) \dot{w}, \zeta_2) \, dr + \tilde{g}(\dot{w}(0), \tilde{\nabla}_{\zeta_1(0)} \zeta_2).$$

Since the inclusions of $H^1([0, t], \mathbb{R}^m)$ into $L^2([0, t], \mathbb{R}^m)$ and into $C^0([0, t], \mathbb{R}^m)$ are compact (see [4]) and keeping in mind (6.27), formula (6.30) tells us that \mathbf{K}_t is a compact operator for every $t \in]0, 1]$. For all t , let $\{\lambda_l(t)\}_{k \in \mathbb{N}}$ be the sequence of all the eigenvalues of \mathbf{K}_t ; they can be characterized by the following *minimax* property:

$$\lambda_l(t) = \max_{\dim(V)=l} \min_{\substack{\xi \in V \\ \|\xi\|_t = 1}} \langle \mathbf{K}_t[\xi], \xi \rangle_t,$$

where the first maximum is taken over all possible subspaces V of \mathbf{T}_t having dimension equal to l .

By standard arguments (see for instance [16]) using the above characterization of the λ_l 's one proves that the map

$$t \longmapsto \lambda_l(t)$$

is continuous.

We now prove the following claims:

1. for t small enough, \mathbf{H}_t is positive definite in \mathbf{T}_t ;
2. for all t , the kernel of \mathbf{H}_t consists precisely of all γ -Jacobi fields along $w|_{[0,t]}$ that vanish at t ;
3. for all $k \in \mathbb{N}$, the map $t \mapsto \lambda_l(t)$ is increasing on $]0, 1[$; moreover, if for some $t_0 \in]0, 1[$ it is $\lambda_l(t_0) = 1$, then $\lambda_l(t) > 1$ for all $t \in]t_0, 1[$.

Observe that the proof will be concluded once the above claims are proven. Indeed, by definition, a point $w(t_0)$ is a γ -focal point along w with multiplicity d if and only if there exists $k > 0$ such that $\lambda_l(t_0) = \lambda_{l+1}(t_0) = \dots = \lambda_{l+d-1}(t_0) = 1$. From (6.29), the Morse index of $H^E(w)$ on $T_w\Omega_{\gamma,p}^{(1)}(\tilde{\Delta})$ is given by the sum of the dimensions of the eigenspaces of \mathbf{K}_1 corresponding to eigenvalues $\lambda_l(1)$ which are strictly larger than one. By the claims 1, 2 and 3 above, such number is given by the sum of the dimensions of the kernels of \mathbf{H}_t , the sum being taken over all $t \in]0, 1[$. By definition, this number is equal to the geometric index $\tilde{\mu}(w)$ of w .

Let's prove the claim 1; observe that another way of stating this claim is that, for all $k \in \mathbb{N}$ and for $t > 0$ small enough, we have:

$$\lambda_l(t) < 1.$$

For $\zeta \in \mathbf{T}_t$, since $\zeta(t) = 0$, we have:

$$\tilde{g}(\zeta(r), \zeta(r)) = -2 \int_r^t \tilde{g}(\zeta, \tilde{\nabla} \zeta) \, dr,$$

hence, using Schwartz's inequality we have:

$$(6.31) \quad \|\zeta(r)\|^2 \leq 2 \int_r^t \|\zeta\| \cdot \|\tilde{\nabla} \zeta\| \, dr \leq 2 \left(\int_0^t \|\zeta\|^2 \, dr \right)^{\frac{1}{2}} \left(\int_0^t \|\tilde{\nabla} \zeta\|^2 \, dr \right)^{\frac{1}{2}}.$$

Integrating (6.31) on $[0, t]$ we obtain:

$$\int_0^t \|\zeta\|^2 \, dr \leq 2t \left(\int_0^t \|\zeta\|^2 \, dr \right)^{\frac{1}{2}} \left(\int_0^t \|\tilde{\nabla} \zeta\|^2 \, dr \right)^{\frac{1}{2}}$$

from which we get:

$$(6.32) \quad \int_0^t g(\tilde{\nabla} \zeta, \tilde{\nabla} \zeta) \, dr \geq \frac{1}{4t^2} \int_0^t \tilde{g}(\zeta, \zeta) \, dr.$$

Moreover, another application of Schwartz's inequality gives us:

$$\|\zeta(0)\| \leq \int_0^t \|\tilde{\nabla} \zeta\| \, dr \leq \sqrt{t} \cdot \left(\int_0^t \|\tilde{\nabla} \zeta\|^2 \, dr \right)^{\frac{1}{2}},$$

from which we obtain the inequality:

$$(6.33) \quad \int_0^t \|\tilde{\nabla} \zeta\|^2 \, dr \geq \frac{1}{t} \|\zeta(0)\|^2.$$

The proof of claim 1 follows immediately from (6.26), (6.27), (6.32) and (6.33).

For the claim 2, we need to show that $\zeta \in \mathbf{T}_t$ and the equality $\mathbf{H}_t(\zeta, \zeta_1) = 0$ holds for all $\zeta_1 \in \mathbf{T}_t$ if and only if ζ is a γ -Jacobi field along w , i.e., if and only if ζ satisfies the four conditions:

$$(6.34) \quad \begin{aligned} &\tilde{\nabla}_w^2 \zeta - \tilde{R}(\dot{w}, \zeta) \dot{w} = 0, \quad \zeta(0) \parallel Y(w(0)), \\ &\zeta(t) = 0, \quad \text{and} \quad \tilde{g}(\tilde{\nabla}_{\dot{w}(0)} \zeta, Y(w(0))) + \tilde{g}(\dot{w}(0), \tilde{\nabla}_{\zeta(0)} Y) = 0. \end{aligned}$$

For the first part of the claim, it suffices to show that if ζ is a vector field along $w|_{[0,t]}$ such that (6.34) holds, then $\zeta \in \mathbf{T}_t$. Indeed, for any vector field ζ that satisfies (6.34), the equality $\mathbf{H}_t(\zeta, \zeta_1) = 0$ is easily verified using integration by parts. Since Y is Killing and w is a geodesic

in the metric \tilde{g} , then the quantity $\tilde{g}(\tilde{\nabla}_{\dot{w}}\zeta, Y) + \tilde{g}(\dot{w}, \tilde{\nabla}_{\zeta}Y)$ is constant along w , hence (6.34) implies that $\zeta \in \mathbf{T}_t$.

Conversely, let's assume that $\mathbf{H}_t(\zeta, \zeta_1) = 0$ for all $\zeta_1 \in \mathbf{T}_t$. Let V be an arbitrary smooth vector field along $w|_{[0,t]}$ such that $V(0) = V(t) = 0$.

Let us set:

$$(6.35) \quad L_V = \tilde{g}(\tilde{\nabla}_{\dot{w}}V, Y) + \tilde{g}(\dot{w}, \tilde{\nabla}_V Y),$$

and

$$(6.36) \quad \zeta_1 = V - \mu \cdot Y,$$

where

$$(6.37) \quad \mu(r) = - \int_r^t \frac{L_V}{\tilde{g}(Y, Y)} \, du.$$

From the definition (6.37) of μ it is easily checked that $\zeta_1 \in \mathbf{T}_t$; we compute as follows:

$$(6.38) \quad \begin{aligned} \mathbf{H}_t(\zeta, \zeta_1) &= \int_0^t \tilde{g}(\tilde{\nabla}_{\dot{w}}\zeta, \tilde{\nabla}_{\dot{w}}V - \mu' \cdot Y - \mu \cdot \tilde{\nabla}_{\dot{w}}Y) \, dr \\ &+ \int_0^t \tilde{g}(\tilde{R}(\dot{w}, \zeta) \dot{w}, V - \mu \cdot Y) \, dr + \mu(0) \cdot \tilde{g}(\dot{w}(0), \tilde{\nabla}_{\zeta(0)}(Y)) = \\ &= \mathbf{H}_t(\zeta, V) - \mathbf{H}_t(\zeta, \mu \cdot Y). \end{aligned}$$

We now show that $\mathbf{H}_t(\zeta, \mu \cdot Y) = 0$. Since Y is Killing, then its restriction to w is a Jacobi field (see [19, Lemma 26, p. 252]), and so it satisfies:

$$(6.39) \quad \tilde{\nabla}_{\dot{w}}^2 Y = \tilde{R}(\dot{w}, Y) \dot{w}.$$

Integration by parts and (6.39) yield:

$$(6.40) \quad \begin{aligned} \int_0^t \mu \cdot \tilde{g}(\tilde{\nabla}_{\dot{w}}\zeta, \tilde{\nabla}_{\dot{w}}Y) \, dr &= \mu \cdot \tilde{g}(\zeta, \tilde{\nabla}_{\dot{w}}Y)|_0^t - \int_0^t \tilde{g}(\zeta, \mu' \cdot \tilde{\nabla}_{\dot{w}}Y + \mu \cdot \tilde{\nabla}_{\dot{w}}^2 Y) \, dr = \\ &= \mu(0) \cdot \tilde{g}(\dot{w}(0), \tilde{\nabla}_{\zeta(0)}Y) - \int_0^t \left[\mu' \cdot \tilde{g}(\zeta, \tilde{\nabla}_{\dot{w}}Y) + \mu \cdot \tilde{g}(\zeta, \tilde{R}(\dot{w}, Y) \dot{w}) \right] \, dr, \end{aligned}$$

where in the last equality we have used the anti-symmetry of the map $(a, b) \rightarrow \tilde{g}(a, \tilde{\nabla}_b Y)$.

By the symmetry of the curvature tensor, we have:

$$\tilde{g}(\tilde{R}(\dot{w}, \zeta) \dot{w}, Y) = \tilde{g}(\zeta, \tilde{R}(\dot{w}, Y) \dot{w})$$

hence, we have:

$$(6.41) \quad \mathbf{H}_t(\zeta, \mu \cdot Y) = \int_0^t \left[\mu' \cdot \tilde{g}(\tilde{\nabla}_{\dot{w}}\zeta, Y) - \mu' \cdot \tilde{g}(\zeta, \tilde{\nabla}_{\dot{w}}Y) \right] \, dr = 0,$$

because $\zeta \in \mathbf{T}_t$ (see formula (6.24)).

If we use the equality $\mathbf{H}_t(\zeta, \zeta_1) = 0$ we get:

$$(6.42) \quad \begin{aligned} 0 &= \mathbf{H}_t(\zeta, \zeta_1) = \mathbf{H}_t(\zeta, V) = \int_0^t \left[\tilde{g}(\tilde{\nabla}_{\dot{w}}\zeta, \tilde{\nabla}_{\dot{w}}V) + \tilde{g}(\tilde{R}(\dot{w}, \zeta) \dot{w}, V) \right] \, dr = \\ &= - \int_0^t \tilde{g}(\tilde{\nabla}_{\dot{w}}^2 \zeta - \tilde{R}(\dot{w}, \zeta) \dot{w}, V) \, dr. \end{aligned}$$

Since (6.42) holds for all smooth vector field V along w vanishing at the endpoints, the fundamental lemma of Calculus of Variations tells us that:

$$\tilde{\nabla}_{\dot{w}}^2 \zeta - \tilde{R}(\dot{w}, \zeta) \dot{w} = 0,$$

which is the first condition in (6.34). The other three conditions of (6.34) are satisfied by any vector field in \mathbf{T}_t , hence claim 2 is proven.

Let's go now to the proof of claim 3. Let's fix $0 < t_1 < t_2$ in $[0, 1]$; we prove first that, for all l , we have:

$$(6.43) \quad \lambda_l(t_1) \leq \lambda_l(t_2).$$

To this goal, let l be fixed and let V_1 be a l -dimensional subspace of \mathbf{T}_{t_1} such that:

$$\lambda_l(t_1) = \min_{\substack{\xi \in V_1 \\ \|\xi\|_{t_1} = 1}} \langle\langle \mathbf{K}_{t_1}[\xi], \xi \rangle\rangle_{t_1}.$$

We define a linear and continuous map $I_{t_1, t_2} : \mathbf{T}_{t_1} \longrightarrow \mathbf{T}_{t_2}$ given by:

$$I_{t_1, t_2}(\xi)(r) = \begin{cases} \xi(r), & \text{if } r \leq t_1; \\ 0, & \text{if } r \in]t_1, t_2]. \end{cases}$$

We observe that, with the above definition, $I_{t_1, t_2}(\xi)$ does indeed belong to \mathbf{T}_{t_2} (see formula 6.24)); observe also that I_{t_1, t_2} is an isometry, and in particular injective. Moreover, the following equality holds trivially:

$$(6.44) \quad \langle\langle \mathbf{K}_{t_1}[\xi], \xi \rangle\rangle_{t_1} = \langle\langle \mathbf{K}_{t_2}[I_{t_1, t_2}(\xi)], I_{t_1, t_2}(\xi) \rangle\rangle_{t_2}, \quad \forall \xi \in \mathbf{T}_{t_1}.$$

Let V_2 be the l -dimensional subspace of \mathbf{T}_{t_2} defined by:

$$V_2 = I_{t_1, t_2}(V_1).$$

Then, by (6.44), we have:

$$(6.45) \quad \begin{aligned} \lambda_l(t_1) &= \min_{\substack{\xi \in V_1 \\ \|\xi\|_{t_1} = 1}} \langle\langle \mathbf{K}_{t_1}[\xi], \xi \rangle\rangle_{t_1} = \min_{\substack{\eta \in V_2 \\ \|\eta\|_{t_2} = 1}} \langle\langle \mathbf{K}_{t_2}[\eta], \eta \rangle\rangle_{t_2} \leq \\ &\leq \max_{\dim(W)=k} \min_{\substack{\eta \in W \\ \|\eta\|_{t_2} = 1}} \langle\langle \mathbf{K}_{t_2}[\eta], \eta \rangle\rangle_{t_2} = \lambda_l(t_2), \end{aligned}$$

which proves (6.43).

To prove the second part of claim 3, it suffices to observe that if $\lambda_l(t_0) = 1$ then $w(t_0)$ is a γ -focal point along w . Since the set of γ -focal points along w is discrete (see Remark 6.2), it follows that, if $\lambda_l(t_0) = 1$, then $\lambda_l(t) \neq 1$ in a neighborhood of t_0 . Finally, by the monotonicity of λ_l , we conclude that $\lambda_l(t) > 1$ in $]t_0, 1]$, and we are done. \square

Theorem 6.9 can be applied to the Riemannian metric $\tilde{g} = \phi_k \cdot g_R$ defined in U_k . Recalling that $\bar{m}(w, E_{\phi_k})$ denotes the Morse Index of the restriction of the Hessian $H^{E_{\phi_k}}$ on the space $T_w \Omega_{\gamma, p}^{(1)}(\Delta)^\perp$, we have thus proven the equality:

$$(6.46) \quad \bar{m}(w, E_{\phi_k}) = m(w, E_{\phi_k}).$$

7. THE INDEX THEOREM FOR BRACHISTOCHRONES

We want to study now the Morse index of the travel time functional at a given brachistochrone σ , which is defined as the index of the symmetric bilinear form $H^T(\sigma)$ (see Definition 6.5).

In this section we extend the classical the Morse Theory for Riemannian geodesics, in order to obtain a weak version of the Morse Index Theorem for brachistochrones (Theorem 7.12), by introducing the concepts of b-Jacobi fields and b-focal points along a brachistochrone σ (see Definitions 7.1 and 7.6 below).

We now begin with the study of the Hessian of the travel time functional.

Let $\sigma \in \mathcal{B}_{p, \gamma}^{(1)}(k)$ be a brachistochrone, since $\mathcal{T}_\sigma > 0$, formula (5.5) tells us that:

$$(7.1) \quad m(\sigma, T) = m(\sigma, -F), \quad \text{and} \quad \text{Ker}(H^T(\sigma)) = \text{Ker}(H^F(\sigma)).$$

We emphasize that from now on we will consider brachistochrone curves whose endpoints may vary in the open set U_k , whereas the value of their energy constant k is a *fixed* positive number. For the sake of shortness, when speaking of brachistochrones we will omit to specify the value of their energy constant without danger of confusion.

In this section and in the rest of the paper we will be speaking of *variations* of a given curve in some fixed space, which will be a family of curves *of the same type*, in a sense that will be clarified in the different situations, parameterized by a suitable variable, denoted by s . Whenever not specified, we will tacitly assume that s varies in an interval of the form $] - \varepsilon, \varepsilon [$ for some $\varepsilon > 0$. A formal definition of smooth variation of a given curve $z \in \Omega_{p,\gamma}^{(1)}$ is given in Appendix A (Definition A.2).

We also warn the reader that, in the course of the section, we will switch back and forth among the three Hessians H^T , H^F and $H^{E_{\phi_k}}$, keeping in mind the basic relations among them given by formulas (5.4) and (5.6).

We mimic the classical Morse theory and we proceed as follows.

Let $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ be a fixed brachistochrone, and, recalling the definition of the space $\mathcal{B}^{(1)}(k)$ given in (2.39), we consider a variation $\sigma_s \in \mathcal{B}^{(1)}(k)$ of σ , depending smoothly on the parameter $s \in] - \varepsilon, \varepsilon [$ and such that $\sigma_0 = \sigma$. Suppose that each curve σ_s is a brachistochrone of energy k between $\sigma_s(0)$ and $\gamma_{\sigma_s(1)}$, where $\gamma_{\sigma_s(1)}$ is the integral line of Y passing through $\sigma_s(1)$.

This means that each σ_s satisfies the differential equation (3.21) and with initial tangent vector $\dot{\sigma}_s(0)$ satisfying the two conditions:

$$(7.2) \quad \langle \dot{\sigma}_s(0), Y(\sigma_s(0)) \rangle^2 + k^2 \langle \dot{\sigma}_s(0), \dot{\sigma}_s(0) \rangle = 0, \quad \text{and} \quad \langle \dot{\sigma}_s(0), Y(\sigma_s(0)) \rangle < 0.$$

Definition 7.1. A vector field $V \in T_\sigma \mathcal{B}^{(1)}(k)$ along the brachistochrone σ in $\mathcal{B}_{p,\gamma}^{(1)}(k)$ is called a *b-Jacobi field* if there exists a variation $\sigma_s \in \mathcal{B}^{(1)}(k)$ of σ as above such that $V = \frac{d}{ds} \Big|_{s=0} \sigma_s$.

In other words, a b-Jacobi field along σ is a variational vector field corresponding to variations made of brachistochrones with the same energy constant and, possibly, with different endpoints. By definition, the b-Jacobi fields are characterized by the property of satisfying the linearized brachistochrone equation; this second order differential equation has a rather ugly aspect and it is presented only for the sake of completeness in the following Proposition.

Proposition 7.2. Let $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ be a brachistochrone of travel time \mathcal{T}_σ and let $V \in T_\sigma \mathcal{B}^{(1)}(k)$ be a variational vector field along σ , with constant $C_V = \langle \nabla_{\dot{\sigma}} V, Y \rangle - \langle V, \nabla_{\dot{\sigma}} Y \rangle$. If V is a b-Jacobi field then V satisfies the second order linear differential equation:

$$(7.3) \quad \begin{aligned} & \nabla_{\dot{\sigma}}^2 V - R(\dot{\sigma}, V) \dot{\sigma} + \frac{2k \mathcal{T}_\sigma}{\langle Y, Y \rangle^2} \left(\nabla_{\dot{\sigma}} \nabla_V Y - \langle Y, Y \rangle R(\dot{\sigma}, V) Y - 2 \langle \nabla_V Y, Y \rangle \nabla_{\dot{\sigma}} Y \right) + \\ & - 2 \frac{C_V}{\langle Y, Y \rangle} \nabla_{\dot{\sigma}} Y + \frac{2k^2 \dot{\sigma} - 2k \mathcal{T}_\sigma Y}{\langle Y, Y \rangle (k^2 + \langle Y, Y \rangle)} \left(\langle \nabla_{\dot{\sigma}} \nabla_V Y, Y \rangle + \langle \nabla_V Y, \nabla_{\dot{\sigma}} Y \rangle \right) + \\ & + \frac{2k^2 \dot{\sigma} - 2k \mathcal{T}_\sigma Y}{\langle Y, Y \rangle^2 (k^2 + \langle Y, Y \rangle)^2} \times \\ & \quad \times \left(- 4 \langle \nabla_{\dot{\sigma}} Y, Y \rangle \langle Y, Y \rangle \langle \nabla_V Y, Y \rangle - 2k^2 \langle \nabla_{\dot{\sigma}} Y, Y \rangle \langle \nabla_V Y, Y \rangle \right) + \\ & + \frac{2 \langle \nabla_{\dot{\sigma}} Y, Y \rangle}{\langle Y, Y \rangle (k^2 + \langle Y, Y \rangle)} \left(C_V Y - k \mathcal{T}_\sigma \nabla_V Y + k^2 \nabla_{\dot{\sigma}} V \right) = 0, \end{aligned}$$

and the initial condition:

$$(7.4) \quad -\mathcal{T}_\sigma C_V + k \langle \nabla_{\dot{\sigma}} V(0), \dot{\sigma}(0) \rangle = 0.$$

Proof. The equation (7.3) is obtained by patiently linearizing the brachistochrone differential equation (3.21), using the following *dictionary*:

- $-k \frac{d}{ds} \Big|_{s=0} (\mathcal{T}_{\sigma_s}) = C_V;$
- $\frac{D}{ds} \Big|_{s=0} (\dot{\sigma}_s) = \nabla_{\dot{\sigma}} V;$
- $\frac{D}{ds} \Big|_{s=0} (\nabla_{\dot{\sigma}_s} \dot{\sigma}_s) = \nabla_{\dot{\sigma}}^2 V - R(\dot{\sigma}, V) \dot{\sigma};$
- $\frac{d}{ds} \Big|_{s=0} (\langle Y(\sigma_s), Y(\sigma_s) \rangle) = 2 \langle \nabla_V Y, Y \rangle;$
- $\frac{D}{ds} \Big|_{s=0} (\nabla_{\dot{\sigma}_s} Y) = \nabla_{\dot{\sigma}} \nabla_V Y + R(V, \dot{\sigma}) Y = \nabla_{\dot{\sigma}} \nabla_V Y - R(\dot{\sigma}, V) Y;$
- $\frac{d}{ds} \Big|_{s=0} (\langle \nabla_{\dot{\sigma}_s} Y, Y \rangle) = \langle \nabla_{\dot{\sigma}} \nabla_V Y, Y \rangle + \langle \nabla_V Y, \nabla_{\dot{\sigma}} Y \rangle;$
- $\frac{d}{ds} \Big|_{s=0} [\langle Y(\sigma_s), Y(\sigma_s) \rangle (k^2 + \langle Y(\sigma_s), Y(\sigma_s) \rangle)] = (4 \langle Y, Y \rangle + 2k^2) \langle \nabla_V Y, Y \rangle.$

The formulas above are obtained by considering the basic properties of the Levi-Civita connection and the curvature tensor of g . In particular, in the sixth formula we have used the fact that $\langle R(\dot{\sigma}, Y) Y, Y \rangle = 0$, by the anti-symmetry in the last two variables.

The initial condition (7.4) is obtained by linearizing the first equation of formula (7.2). \square

A partial converse to Proposition 7.2 is provided by the following Proposition:

Proposition 7.3. *Let $\sigma \in \mathcal{B}^{(1)}(k)$ be a brachistochrone and suppose that V is a smooth vector field along σ satisfying the differential equation (7.3), the initial condition (7.4) and with $V(0) = 0$. Then, V is a b-Jacobi field along σ , i.e., there exists a variation σ_s of σ consisting of brachistochrones between p and γ_s , $s \in]-\varepsilon, \varepsilon[$, such that $V = \frac{d}{ds} \Big|_{s=0} \sigma_s$.*

Proof. We use a sort of *brachistochrone exponential map*, as follows.

Given a vector $v_0 \in T_p \mathcal{M}$ such that

$$(7.5) \quad \langle v_0, Y(p) \rangle^2 + k^2 \langle v_0, v_0 \rangle = 0, \quad \text{and} \quad \langle v_0, v_0 \rangle < 0,$$

then there exists a unique brachistochrone $\sigma_{v_0} \in \mathcal{B}_p^{(1)}(k)$ and such that $\dot{\sigma}_{v_0}(0) = v_0$. This is obtained by solving the differential equation (3.21) with initial conditions $\sigma(0) = p$ and $\dot{\sigma}(0) = v_0$.

Moreover, the map $v_0 \mapsto \sigma_{v_0} \in \mathcal{B}_p^{(1)}(k)$ is C^1 , due to the regular dependence on the data of the solution of the differential equation (3.21).

Let $S \subset T_p \mathcal{M}$ be the set of vectors v_0 satisfying the conditions (7.5); S is a submanifold of $T_p \mathcal{M}$. Indeed, the condition $\langle v_0, v_0 \rangle < 0$ is open; moreover, the gradient of the smooth map $G : T_p \mathcal{M} \ni v_0 \mapsto \langle v_0, Y(p) \rangle^2 + k^2 \langle v_0, v_0 \rangle \in \mathbb{R}$ is easily computed as:

$$(7.6) \quad G'(v_0) = 2 \langle v_0, Y(p) \rangle \cdot Y(p) + 2k^2 v_0.$$

Multiplying by $Y(p)$ we obtain:

$$\langle G'(v_0), Y(p) \rangle = 2 \langle v_0, Y(p) \rangle (\langle Y(p), Y(p) \rangle + k^2) \neq 0,$$

where the last inequality depends on the fact that both v_0 and $Y(p)$ are timelike, hence $\langle v_0, Y(p) \rangle \neq 0$, and $\langle Y(p), Y(p) \rangle + k^2 > 0$ in U_k . This implies that $G' \neq 0$, hence $G^{-1}(0)$ is a smooth submanifold of $T_p \mathcal{M}$. Clearly, $\dot{\sigma}(0) \in S$.

Let $v_0(s) :]-\varepsilon, \varepsilon[\rightarrow S$ be a smooth map such that $v_0(0) = \dot{\sigma}(0) \in S$ and $v'_0(0) = \nabla_{\dot{\sigma}(0)} V$. Observe that $\nabla_{\dot{\sigma}(0)} V$ belongs to $T_{\dot{\sigma}(0)} S$, because, from (7.6), we have:

$$\langle G'(\dot{\sigma}(0)), \nabla_{\dot{\sigma}(0)} V \rangle = 2 \langle \dot{\sigma}(0), Y(p) \rangle \langle Y(p), \nabla_{\dot{\sigma}(0)} V \rangle + 2k^2 \langle \dot{\sigma}(0), \nabla_{\dot{\sigma}(0)} V \rangle.$$

Since $V(0) = 0$, then $C_V = \langle Y(p), \nabla_{\dot{\sigma}(0)} V \rangle$, so we have:

$$\langle G'(\dot{\sigma}(0)), \nabla_{\dot{\sigma}(0)} V \rangle = 2k (-\mathcal{T}_\sigma C_V + k \langle \dot{\sigma}(0), \nabla_{\dot{\sigma}(0)} V \rangle) = 0,$$

where the last equality follows immediately from (7.4). Hence, $\nabla_{\dot{\sigma}(0)}V \in T_{\dot{\sigma}(0)}S$ and the curve $v_0(s)$ is well defined.

Now, for all $s \in]-\varepsilon, \varepsilon[$, let σ_s be the unique brachistochrone in $\mathcal{B}_p^{(1)}(k)$ satisfying $\dot{\sigma}_s(0) = v_0(s)$; clearly, $\sigma_0 = \sigma$, and σ_s is a smooth variation of σ . Observe that, since σ_0 is defined on the closed interval $[0, 1]$, then we can assume that also σ_s is defined on $[0, 1]$ for all s .

In order to conclude the proof, we need to show that the variational field $\tilde{V} = \frac{d}{ds}\Big|_{s=0} \sigma_s$ coincides with V .

By Proposition 7.2, \tilde{V} satisfies the second order differential equation (7.3), while V satisfies (7.3) by assumption, and $\tilde{V}(0) = V(0) = 0$, because we are fixing the initial point p . By uniqueness, in order to prove that $\tilde{V} = V$ along σ it suffices to show that $\nabla_{\dot{\sigma}(0)}\tilde{V} = \nabla_{\dot{\sigma}(0)}V$. This is easily established by the following calculation, that concludes the proof:

$$\nabla_{\dot{\sigma}(0)}\tilde{V} = \frac{D}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \sigma_s = \frac{D}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} \sigma_s = \frac{D}{ds}\Big|_{s=0} \dot{\sigma}_s(0) = v'_0(0) = \nabla_{\dot{\sigma}(0)}V.$$

□

Corollary 7.4. *If σ is a brachistochrone and V is a b-Jacobi field along σ such that $V(0) = 0$, then $V \in T_\sigma \mathcal{B}_p^{(1)}(k)$.*

Proof. Following the proof of Proposition 7.3, V is the variational vector field corresponding to a variation $\sigma_s \in \mathcal{B}_p^{(1)}(k)$ of σ . □

In general, it may not be true that a b-Jacobi field V along a brachistochrone σ satisfying $V(0) = 0$ and $V(1) \in \mathbb{R} \cdot Y(\sigma(1))$ is the variational vector field corresponding to a family of brachistochrones in $\mathcal{B}_{p,\gamma}^{(1)}(k)$. However, such vector fields belong to the tangent space $T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$, and they are in the kernel of the Hessian $H^F(\sigma)$:

Corollary 7.5. *If $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ is a brachistochrone and V is a b-Jacobi field along σ such that $V(0) = 0$ and $V(1)$ is parallel to $Y(\sigma(1))$, then $V \in T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$, and $V \in \text{Ker}(H^F(\sigma))$.*

Proof. By Corollary 7.4, $V \in T_\sigma \mathcal{B}_p^{(1)}(k)$; the first part of the statement follows immediately by observing that a vector field $V \in T_\sigma \mathcal{B}_p^{(1)}(k)$ belongs to $T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$ if and only if $V(1)$ is parallel to $Y(\sigma(1))$ (see formulas (2.11), (2.44) and (2.45)).

To prove the second part of the thesis, we need to show that $H^F(\sigma)[V, W] = 0$ for all $W \in T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$. By Corollary 5.2, we have:

$$(7.7) \quad H^F(\sigma)[V, W] = -H^{E_{\phi_k}}(\mathcal{D}(\sigma))[\text{d}\mathcal{D}(\sigma)[V], \text{d}\mathcal{D}(\sigma)[W]],$$

hence, to conclude the proof it suffices to show that $\text{d}\mathcal{D}(\sigma)[V]$ is in the kernel of the Hessian $H^{E_{\phi_k}}(\mathcal{D}(\sigma))$. By (6.19), this amounts to proving that $X = \text{d}\mathcal{D}(\sigma)[V]$ is the variational vector field corresponding to a smooth variation w_s of $w = \mathcal{D}(\sigma)$ consisting of horizontal geodesics in the metric $\phi_k \cdot g_R$ between p and some integral curve γ_s of Y lying in U_k (recall that a vector field along a geodesic is Jacobi if and only if it is the variational vector field corresponding to a variation by geodesics).

To see this, let σ_s be a smooth variation of σ consisting of brachistochrones in $\mathcal{B}_p^{(1)}(k)$ between p and some curve γ_s in U_k , and with variational vector field V . Such a variation exists by Proposition 7.3.

Then, if we consider the curves $w_s = \mathcal{D}(\sigma_s)$, by part 3 of Proposition 4.5, each w_s is a horizontal geodesic between p and γ_s ; by Proposition 4.4, w_s is a smooth variation of w . Finally, we have:

$$\frac{d}{ds}\Big|_{s=0} w_s = \frac{d}{ds}\Big|_{s=0} \mathcal{D}(\sigma_s) = \text{d}\mathcal{D}(\sigma)\Big[\frac{d}{ds}\Big|_{s=0} \sigma_s\Big] = \text{d}\mathcal{D}(\sigma)[V] = X,$$

which concludes the proof. □

We will see later (Proposition 7.11) that the kernel of the Hessian $H^F(\sigma)$ consists precisely of the b-Jacobi fields along σ ; this fact can also be checked directly using the explicit formula for the Hessian $H^F(\sigma)$ given in Appendix A and the Lagrange multipliers technique.

We are now ready to define the notion of a b-focal point along a brachistochrone.

Definition 7.6. Let $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ be a brachistochrone. A point $\sigma(t_0)$ of σ is said to be a *b-focal point* if there exists a non zero b-Jacobi field V along $\sigma|_{[t_0,1]}$ that vanish at t_0 , that is, a non zero vector field V along σ for which the quantity $C_V = \langle \nabla_{\dot{\sigma}} V, Y \rangle - \langle V, \nabla_{\dot{\sigma}} Y \rangle$ is constant along σ , such that $V(t_0) = 0$, satisfying the differential equation (7.3) and the condition:

$$(7.8) \quad -\mathcal{T}_{\sigma} C_V + k \langle \nabla_{\dot{\sigma}} V(t_0), \dot{\sigma}(t_0) \rangle = 0.$$

In the above situation, we will also say that $\sigma(t_0)$ is *b-conjugate* to $\sigma(1) = p$ along σ .

For every $t_0 \in [0, 1]$, the set $\mathcal{J}_{\sigma}(t_0)$ of vector fields V satisfying the above conditions in the interval $[t_0, 1]$ is a vector field; if $\sigma(t_0)$ is a b-focal point along σ , then *multiplicity* $\mu_{\sigma}(t_0)$ of $\sigma(t_0)$ is the dimension of $\mathcal{J}_{\sigma}(t_0)$. The *geometric index* $\mu(\sigma)$ of the brachistochrone σ is defined to be the (possibly infinite) number:

$$(7.9) \quad \mu(\sigma) = \sum_{t_0 \in [0,1[} \mu_{\sigma}(t_0) \in \mathbb{N} \cup \{+\infty\}.$$

Observe that every vector field along $\sigma|_{[t_0,1]}$ which is solution of the linear differential equation (7.3) in the interval $[t_0, 1]$, can be extended to a vector field along σ satisfying the equation on the entire interval $[0, 1]$. Also, it follows easily from Propositions 4.1 and 7.3 that if the quantity $\langle \nabla_{\dot{\sigma}} V, Y \rangle - \langle V, \nabla_{\dot{\sigma}} Y \rangle$ is constant on $[t_0, 1]$ and if V satisfies (7.3) on $[0, 1]$, then $\langle \nabla_{\dot{\sigma}} V, Y \rangle - \langle V, \nabla_{\dot{\sigma}} Y \rangle$ is constant on $[0, 1]$. In particular, from Proposition 7.3 we have that $\sigma(t_0)$ is a b-focal point if and only if there exists a non trivial variation σ_s , $s \in]-\varepsilon, \varepsilon[$ of brachistochrones of energy k between $\sigma(t_0)$ and γ , depending smoothly on s , and such that $\sigma_0 = \sigma|_{[t_0,1]}$.

We now want to relate the b-focal points along a brachistochrone σ with the γ -focal points along the corresponding Riemannian geodesic $w = \mathcal{D}(\sigma)$. This is done in Theorem 7.12 below, which is preceded by some preliminary results, aimed to determine the relation of the notions of Jacobi fields along σ and w .

More precisely, we will show that the linear map $d\mathcal{O} \circ d\mathcal{D}(\sigma)$ gives an isomorphism of the spaces $\mathcal{J}_{\sigma}(t_0)$ and $\mathcal{J}_w^{\{k\}}(\gamma, t_0)$ (recall that the map \mathcal{O} is the direction reversing map defined in (6.1)).

Given a horizontal geodesic w , a Jacobi field along w is a (smooth) vector field J along w satisfying the differential equation (6.6). From (2.11) and (4.5), such a vector field J belongs to the tangent space $T_w \Omega_{\gamma,p}^{(1)}(\Delta)$ if and only if $J(1) = 0$, $J(0) \in \mathbb{R} \cdot Y(w(0))$ (recall that we are considering curves w starting on γ and arriving at p), and $\langle \nabla_{\dot{w}} J, Y \rangle + \langle \dot{w}, \nabla_J Y \rangle \equiv 0$. Recalling Remark 6.3, this last equality is satisfied identically on $[0, 1]$ provided that it is satisfied at some point $t_0 \in [0, 1]$.

Hence, recalling the definitions 1, 2 and 3 of page 25 and Remark 6.3, we have that the set of Jacobi fields in $T_w \Omega_{\gamma,p}^{(1)}(\Delta)$ coincides with the finite dimensional vector space $\mathcal{J}_w^{\{k\}}(\gamma, 0)$:

$$(7.10) \quad \mathcal{J}_w^{\{k\}} \cap T_w \Omega_{\gamma,p}^{(1)}(\Delta) = \mathcal{J}_w^{\{k\}}(\gamma, 0).$$

We introduce the following map:

$$(7.11) \quad \mathcal{G} : \Omega_{p,\gamma}^{(1)} \longrightarrow \Omega_{p,\gamma}^{(1)},$$

given by:

$$(7.12) \quad \mathcal{G}(w)(t) = \psi(w(t), h_w(t)),$$

where

$$h_w(t) = -k \int_0^t \frac{\sqrt{\phi_k(w(0)) \langle \dot{w}(0), \dot{w}(0) \rangle_{(R)}}}{\langle Y, Y \rangle} dr.$$

As in the case of the map \mathcal{D} , it is easy to see that \mathcal{G} is smooth; moreover, using (4.19) one checks that it is a left-inverse for \mathcal{D} in $\mathcal{B}_{p,\gamma}^{(1)}(k)$, i.e., for all $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$, we have:

$$(7.13) \quad \mathcal{G}(\mathcal{D}(\sigma)) = \sigma.$$

Proposition 7.7. *Let σ be a brachistochrone and $w = \mathcal{O}(\mathcal{D}(\sigma))$. If $J \in \mathcal{J}_w^{\{k\}}(\gamma, 1)$, then there exists $V \in \mathcal{J}_\sigma(0)$ a b-Jacobi field along σ such that $d\mathcal{O} \circ d\mathcal{D}(\sigma)[V] = J$.*

Proof. Let $s \in]-\varepsilon, \varepsilon[$ and w_s be a smooth variation of w consisting of horizontal geodesics and such that $J = \frac{d}{ds} \Big|_{s=0} w_s$. Let $\sigma_s = \mathcal{G}(\mathcal{O}(w_s)) \in \mathcal{B}_p^{(1)}(k)$; since \mathcal{G} is smooth, then σ_s is a smooth variation of σ . Moreover, $\mathcal{O}(\mathcal{D}(\sigma_s)) = w_s$, and since w_s is a horizontal geodesic, by Proposition 4.5, σ_s is a brachistochrone in $\mathcal{B}_p^{(1)}(k)$ for all s . By Definition 7.1, $V = \frac{d}{ds} \Big|_{s=0} \sigma_s$ is a b-Jacobi field in $\mathcal{J}_\sigma(0)$. Note that $V(0) = 0$ because $\sigma_s(0) = p$ for all s .

It is easily computed:

$$d\mathcal{O} \circ d\mathcal{D}(\sigma)[V] = \frac{d}{ds} \Big|_{s=0} \mathcal{O}(\mathcal{D}(\sigma_s)) = \frac{d}{ds} \Big|_{s=0} w_s = J,$$

which concludes the proof. \square

Proposition 7.7 gives the surjectivity of the map $d\mathcal{O} \circ d\mathcal{D}(\sigma)$ restricted to the spaces of Jacobi fields $\mathcal{J}_\sigma(0)$ and $\mathcal{J}_w^{\{k\}}(\gamma, 0)$. The injectivity of $d\mathcal{D}(\sigma)$, and hence that of $d\mathcal{O} \circ d\mathcal{D}(\sigma)$, can be proven on the entire tangent space $T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$:

Proposition 7.8. *For all $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$, $d\mathcal{D}(\sigma) : T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k) \mapsto T_{\mathcal{D}(\sigma)} \Omega_{p,\gamma}^{(1)}$ is an injective map.*

Proof. It suffices to prove that $d\mathcal{D}(\sigma)$ has a left inverse, i.e., that there exists a linear bounded operator $L : T_{\mathcal{D}(\sigma)} \Omega_{p,\gamma}^{(1)} \mapsto T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$ such that $L \circ d\mathcal{D}(\sigma)$ is the identity on $T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$. Such a map L is given by the differential of the map \mathcal{G} defined by (7.12). Indeed, by (7.13), $\mathcal{G} \circ \mathcal{D}$ is the identity on $\mathcal{B}_{p,\gamma}^{(1)}(k)$, and by differentiating we have that $d\mathcal{G} \circ d\mathcal{D}(\sigma)$ is the identity on $T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$ for all $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$. \square

We can indeed identify the image of $d\mathcal{D}(\sigma)$ in $T_{\mathcal{D}(\sigma)} \Omega_{p,\gamma}^{(1)}$:

Proposition 7.9. *Let $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ be a brachistochrone and $w = \mathcal{D}(\sigma)$. Then, the image of the differential $d\mathcal{D}(\sigma)$ in $T_w \Omega_{p,\gamma}^{(1)}$ is given by $T_w \Omega_{p,\gamma}^{(1)}(\Delta)^\perp$ (see formula (6.3)).*

Proof. We first show that $d\mathcal{D}(\sigma) \subset T_w \Omega_{p,\gamma}^{(1)}(\Delta)^\perp$. To this end, let $\zeta \in T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$ be fixed; by (2.44) and Corollary 2.4, it satisfies:

$$(7.14) \quad \langle \nabla_{\dot{\sigma}} \zeta, \dot{\sigma} \rangle \equiv 0.$$

Since $\mathcal{D}(\mathcal{B}_{p,\gamma}^{(1)}(k)) \subset \Omega_{p,\gamma}^{(1)}(\Delta)$, then clearly $d\mathcal{D}(T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)) \subset T_w \Omega_{p,\gamma}^{(1)}(\Delta)$. Moreover, let $V = d\mathcal{D}(\sigma)[\zeta]$. For the inclusion $d\mathcal{D}(T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)) \subset T_w \Omega_{p,\gamma}^{(1)}(\Delta)^\perp$ we need to show that (6.5) is satisfied. Using formulas (3.5), (4.10), (4.11), (4.14), (4.16) and (4.19), we compute easily:

$$(7.15) \quad \begin{aligned} & \langle \nabla \phi_k(w), V \rangle \langle \dot{w}, \dot{w} \rangle + 2 \phi_k(w) \cdot \langle \nabla_{\dot{w}} V, \dot{w} \rangle = \\ & = - \frac{2k^2 \mathcal{T}_\sigma^2}{\langle Y, Y \rangle (k^2 + \langle Y, Y \rangle)} \langle \nabla_Y Y, \zeta \rangle + \frac{2k^2 \mathcal{T}_\sigma^2}{\langle Y, Y \rangle (k^2 + \langle Y, Y \rangle)} \langle \nabla_Y Y, \zeta \rangle = 0. \end{aligned}$$

For the opposite inclusion, we argue as follows. Let V be fixed in $T_w \Omega_{p,\gamma}^{(1)}(\Delta)^\perp$ and let $w_s \in \Omega_p^{(1)}$ be a variation of w with variational vector field V such that $\langle \dot{w}_s, Y(w_s) \rangle \equiv 0$ and $\langle \dot{w}_s, \dot{w}_s \rangle \equiv c_s$

(constant). Such a variation exists,² provided that we do not require the condition $w_s(1) \in \gamma(\mathcal{R})$.

For all s , define $\sigma_s = \mathcal{G}(w_s)$ where \mathcal{G} is the map defined in (7.12). Then, σ_s is a variation of σ in $\mathcal{B}_p^{(1)}(k)$; if $\zeta = \frac{d}{ds}\big|_{s=0} \sigma_s \in T_\sigma \mathcal{B}_p^{(1)}(k)$ is the corresponding variational vector field, then clearly $d\mathcal{D}(\sigma)[\zeta] = V$. To conclude the proof, we need to show that $\zeta \in T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$, i.e., that $\zeta(1)$ is parallel to $Y(\sigma(1))$. Recalling (4.14), this follows easily from the fact that $V(1)$ is a multiple of $Y(w(1))$ and from formula (4.14). This concludes the proof. \square

In analogy with formula (4.14), for all $a \in [0, 1[$ we can define a linear map \mathcal{L}_a on the space of vector fields along $\sigma|_{[a,1]}$ satisfying the two conditions appearing in (2.44) on the interval $[a, 1]$, and taking values in the space of vector fields along $w|_{[a,1]}$.

The map \mathcal{L}_a is given by:

$$(7.16) \quad \mathcal{L}_a[\zeta](r) = d_x \psi(\sigma(r), t_\sigma^a(r))[\zeta(r) + \tau_\zeta^a \cdot Y(\sigma(r))],$$

where

$$\tau_\sigma^a(r) = - \int_a^r \frac{\langle \dot{\sigma}, Y \rangle}{\langle Y, Y \rangle} du, \quad \text{and} \quad \tau_\zeta^a(r) = - \int_a^r \frac{C_\zeta \langle Y, Y \rangle + 2k \mathcal{I}_\sigma \langle \nabla_\zeta Y, Y \rangle}{\langle Y, Y \rangle^2} du.$$

In particular, $\mathcal{L}_0 = d\mathcal{D}(\sigma)$; observe also that, if $\zeta(a) = 0$, then $\mathcal{L}_a[\zeta](a) = 0$.

The result of Propositions 7.7 and 7.8 can be extended immediately to the maps $d\mathcal{O} \circ \mathcal{L}_{t_0} : \mathcal{J}_\sigma(t_0) \mapsto \mathcal{J}_w^{\{k\}}(\gamma, t_0)$ for all $t_0 \in [0, 1[$:

Corollary 7.10. *Let $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ be a brachistochrone and $w = \mathcal{D}(\sigma)$ the corresponding geodesic in $\Omega_{p,\gamma}^{(1)}(\Delta)$. Then, for all $t_0 \in [0, 1[$, the linear map $d\mathcal{O} \circ \mathcal{L}_{t_0}$ gives an isomorphism of the vector spaces of Jacobi fields $\mathcal{J}_\sigma(t_0)$ and $\mathcal{J}_w^{\{k\}}(\gamma, t_0)$.*

Proof. The proofs of Propositions 7.7 and 7.8 can be repeated *verbatim*, by replacing the initial point p with the point $\sigma(t_0)$. The only technical subtlety to worry about is that, when replacing the initial point, it will not hold, in general, that $\sigma(t_0) = w(t_0)$. Nevertheless, this fact is not essential, because one can always reduce to this case by considering a suitable isometry of U_k given by $x \mapsto \psi(x, \bar{t})$. \square

We now prove that the kernel of the Hessian $H^F(\sigma)$ in $T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$ consists precisely of b-Jacobi fields. This gives an analytical characterization of the b-Jacobi fields along a brachistochrone.

Proposition 7.11. *Let σ be a brachistochrone. A vector field $V \in T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$ is a b-Jacobi field along σ if and only if $V \in \text{Ker}(H^F(\sigma))$ in $T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$.*

Proof. Corollary 7.5 proves that any b-Jacobi field along σ is in the kernel of $H^F(\sigma)$.

Conversely, let σ be a fixed brachistochrone and $\zeta \in \text{Ker}(H^F(\sigma))$. From Corollary 7.10, it suffices to prove that the vector field $J = d\mathcal{O} \circ d\mathcal{D}(\sigma)[\zeta]$ is a γ -Jacobi field with respect to the Riemannian metric $\phi_k \cdot g_R$ along the geodesic $w = \mathcal{O}(\mathcal{D}(\sigma))$. Moreover, since $J \in T_w \Omega_{\gamma,p}^{(1)}(\Delta)$, from Lemma 6.4 it suffices to show that J is a Jacobi field along w , i.e., that it satisfies equation (6.6). Observe that, by Proposition 7.9, J is in $T_w \Omega_{\gamma,p}^{(1)}(\Delta)^\perp$, hence it satisfies the two equations:

$$(7.17) \quad \begin{aligned} \langle \nabla_{\dot{w}}^{\{k\}} J, Y \rangle_{(R)} - \langle J, \nabla_{\dot{w}}^{\{k\}} Y \rangle_{(R)} &= 0, \\ \langle J, \dot{w} \rangle_{(R)} &= \langle \nabla_{\dot{w}}^{\{k\}} J, \dot{w} \rangle_{(R)} = 0. \end{aligned}$$

²the point here is that the variational fields in $T_w \Omega_{p,\gamma}^{(1)}$ are given by variations w_s of w that *not necessarily* have endpoints on $\gamma(\mathcal{R})$. The only thing that can be said about such variations w_s is that $w_s(1)$ is infinitesimally close to γ as $s \rightarrow 0$ with an order of infinitesimal bigger than 1.

To prove that J is Jacobi, let $V \in C_o^\infty([0, 1], T\mathcal{M})$ be any smooth vector field along w vanishing at the endpoints. We set:

$$(7.18) \quad W = V - \mu \cdot Y - \lambda \cdot \dot{w},$$

where λ and μ are functions in $H^1([0, 1], \mathbb{R})$ to be determined in such a way that the resulting vector field W belongs to $T_w\Omega_{\gamma,p}^{(1)}(\Delta)$. Straightforward computations show this condition is satisfied by setting:

$$(7.19) \quad \begin{aligned} \mu(t) &= - \int_t^1 \phi_k(w) \cdot \frac{\langle \nabla_{\dot{w}}^{\{k\}} V, Y \rangle_{(R)} + \langle \dot{w}, \nabla_V^{\{k\}} Y \rangle_{(R)}}{\langle Y, Y \rangle_{(R)}} dr, \\ \lambda(t) &= - \int_t^1 \frac{\langle \nabla_{\dot{w}}^{\{k\}} V, \dot{w} \rangle_{(R)}}{\langle \dot{w}, \dot{w} \rangle_{(R)}} dr. \end{aligned}$$

Observe that, with the definitions above, since w is a geodesic with respect to $\phi_k \cdot g_R$ one has:

$$(7.20) \quad \lambda(0) = \lambda(1) = \mu(1) = 0.$$

Arguing as in the proof of Theorem 6.9 since Y is Killing in the metric $\phi_k \cdot g_R$, then its restriction to w is a Jacobi field (see also (6.39)):

$$(7.21) \quad \nabla_{\dot{w}}^{\{k\}} \nabla_{\dot{w}}^{\{k\}} Y = R^{\{k\}}(\dot{w}, Y) \dot{w}.$$

Recalling (6.12), keeping in mind (7.20) and the fact that $V(0) = V(1) = 0$, we have:

$$(7.22) \quad \begin{aligned} H^{E_{\phi_k}}(w)[J, W] &= I^{\{k\}}(J, V) - I^{\{k\}}(J, \lambda \cdot \dot{w}) - I^{\{k\}}(J, \mu \cdot Y) \\ &\quad - \mu(1) \phi_k(w(1)) \langle \nabla_{J(1)}^{\{k\}} Y, \dot{w}(1) \rangle_{(R)}. \end{aligned}$$

From (6.9), the second equation of (7.17) and the anti-symmetry of the curvature tensor $R^{\{k\}}$, the term $I^{\{k\}}(J, \lambda \cdot \dot{w})$ is easily seen to vanish:

$$(7.23) \quad I^{\{k\}}(J, \lambda \cdot \dot{w}) = \int_0^1 \phi_k(w) \left(\lambda' \langle \nabla_{\dot{w}}^{\{k\}} J, \dot{w} \rangle_{(R)} + \lambda \langle R^{\{k\}}(\dot{w}, J) \dot{w}, \dot{w} \rangle_{(R)} \right) dt = 0.$$

From (6.9), integrating by parts and using formulas (7.17), (7.21) and the symmetry of the curvature tensor $R^{\{k\}}$, we have:

$$\begin{aligned}
(7.24) \quad I^{\{k\}}(J, \mu \cdot Y) &= \int_0^1 \phi_k(w) \left(\mu' \cdot \langle \nabla_{\dot{w}}^{\{k\}} J, Y \rangle_{(R)} + \mu \cdot \langle \nabla_{\dot{w}}^{\{k\}} J, \nabla_{\dot{w}}^{\{k\}} Y \rangle_{(R)} \right) dt \\
&\quad + \int_0^1 \phi_k(w) \mu \cdot \langle R^{\{k\}}(\dot{w}, J) \dot{w}, Y \rangle_{(R)} dt = \\
&= \int_0^1 \phi_k(w) \left(\mu' \cdot \langle \nabla_{\dot{w}}^{\{k\}} J, Y \rangle_{(R)} + \mu \cdot \langle R^{\{k\}}(\dot{w}, J) \dot{w}, Y \rangle_{(R)} \right) \\
&\quad - \int_0^1 \phi_k(w) \mu' \cdot \langle J, \nabla_{\dot{w}}^{\{k\}} Y \rangle_{(R)} dt \\
&\quad - \int_0^1 \phi_k(w) \mu \cdot \langle J, \nabla_{\dot{w}}^{\{k\}} \nabla_{\dot{w}}^{\{k\}} Y \rangle_{(R)} dt \\
&\quad + \mu(1) \cdot \phi_k(w(1)) \cdot \langle J(1), \nabla_{\dot{w}(1)}^{\{k\}} Y \rangle_{(R)} = \\
&= \int_0^1 \phi_k(w) \mu' \left(\langle \nabla_{\dot{w}}^{\{k\}} J, Y \rangle_{(R)} - \langle J, \nabla_{\dot{w}}^{\{k\}} Y \rangle_{(R)} \right) dt \\
&\quad + \int_0^1 \phi_k(w) \mu \left(\langle R^{\{k\}}(\dot{w}, J) \dot{w}, Y \rangle_{(R)} - \langle R^{\{k\}}(\dot{w}, Y) \dot{w}, J \rangle_{(R)} \right) dt \\
&\quad + \mu(1) \cdot \phi_k(w(1)) \cdot \langle J(1), \nabla_{\dot{w}(1)}^{\{k\}} Y \rangle_{(R)} = \\
&= -\mu(1) \cdot \phi_k(w(1)) \cdot \langle \dot{w}(1), \nabla_{J(1)}^{\{k\}} Y \rangle_{(R)}.
\end{aligned}$$

Finally, from (7.22), (7.23) and (7.24), we have proven the equality:

$$I^{\{k\}}(J, V) = H^{E_{\phi_k}}(w)[J, W].$$

Since $W \in T_w \Omega_{\gamma, p}^{(1)}(\Delta)$, then W is in the image of $d\mathcal{O} \circ d\mathcal{D}$, say $W = d\mathcal{O} \circ d\mathcal{D}(\sigma)[\zeta_1]$ for some $\zeta_1 \in T_{\sigma} \mathcal{B}_{p, \gamma}^{(1)}(k)$. Since $\zeta \in \text{Ker}(H^F(\sigma))$ and $J = d\mathcal{O} \circ d\mathcal{D}(\sigma)[\zeta]$, then, by Corollary 5.2 and formula (6.2), it is $H^{E_{\phi_k}}(w)[J, W] = -H^F(\sigma)[\zeta, \zeta_1] = 0$, and, in particular, $I^{\{k\}}(J, V) = 0$. Hence, we have that $I^{\{k\}}(J, V) = 0$ for all smooth vector field along w vanishing at the endpoints, and by (6.10) this implies that J is a Jacobi field, concluding the proof. \square

We are finally ready to state and prove the Morse Index Theorem for the travel time brachistochrones:

Theorem 7.12 (Morse Index Theorem for Relativistic Brachistochrones).

Let $\sigma \in \mathcal{B}_{p, \gamma}^{(1)}(k)$ be a brachistochrone and $w = \mathcal{O}(\mathcal{D}(\sigma)) \in \Omega_{\gamma, p}^{(1)}$ the corresponding horizontal geodesic. Then, a point $\sigma(t_0)$ is a b -focal point along σ if and only if $w(t_0)$ is a γ -focal point along w , in which case the two focal points have the same multiplicity. In particular, we have

$$(7.25) \quad \mu(\sigma) = \mu^{\{k\}}(w).$$

Moreover, if p is not a b -focal point along σ , then the Morse index $m(\sigma, T)$ is equal to the geometric index $\mu(\sigma)$ of σ :

$$(7.26) \quad m(\sigma, T) = \mu(\sigma).$$

Proof. By Corollary 7.10, since isomorphisms preserve dimensions, for all $t_0 \in [0, 1[$ we have:

$$\dim \left(\mathcal{J}_w^{\{k\}}(t_0) \right) = \mu_{\sigma}(t_0).$$

This implies that $\sigma(t_0)$ is a b -focal point along σ if and only if $w(t_0)$ is a γ -focal point; moreover, summing over all $t_0 \in [0, 1[$, we obtain (7.25).

From Corollary 5.2 and formulas (5.6) and (6.2), we have:

$$(7.27) \quad m(\sigma, T) = m(\sigma, -F);$$

from (7.7) and Propositions 7.8 and 7.9 we obtain:

$$(7.28) \quad m(\sigma, -F) = \bar{m}(w, E_{\psi_k});$$

finally, from (6.46) we have the equality:

$$(7.29) \quad \bar{m}(w, E_{\phi_k}) = m(w, E_{\phi_k}).$$

If p is not a γ -focal point along w , or equivalently if p is not a b-focal point along σ , then, Theorem 6.8 implies:

$$(7.30) \quad m(w, E_{\phi_k}) = \mu^{\{k\}}(w);$$

the equality (7.26) follows at once from (7.25) and (7.27)—(7.30). This concludes the proof. \square

From finiteness of the index $m(\sigma, T)$ we get the following:

Corollary 7.13. *Let $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ be a brachistochrone. Then, σ is never a local maximum for T .* \square

From the equality (7.26) we get that, if $\mu(\sigma) = 0$, then the Morse index of the travel time vanishes at σ , hence σ is a local minimum for T . Therefore, we have:

Corollary 7.14. *Let $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ be a brachistochrone and $w = \mathcal{O}(\mathcal{D}(\sigma))$. Suppose that there are no γ -focal points along w . Then, σ is a local minimum for the arrival time functional T .* \square

8. THE GLOBAL MORSE RELATIONS

In this section we will use the infinite dimensional Morse theory to prove some equalities relating the differential structure of the travel time brachistochrone problem and the topological structure carried by the set of continuous paths joining p and γ in U_k .

Most of the technical results needed are obtained using the same ideas and techniques employed in Reference [5], where the authors prove the Morse relations for geodesics in a convex subset of a stationary Lorentzian manifold. In order to keep our exposition short, we will omit some of the proofs that can be deduced easily from analogous proofs presented in details in [5].

Throughout the section, we will make the following assumptions:

1. the vector field Y is complete in U_k , i.e., its integral lines are defined over the entire real line;
2. $\gamma : \mathbb{R} \rightarrow U_k$ is an integral line of Y without self-intersection;
3. p is an event in U_k ;
4. k^2 is a regular value for the function $-\langle Y, Y \rangle$;
5. $\bar{U}_k = U_k \cup \partial U_k$ is complete with respect to the Riemannian metric (2.2);
6. the function $-\langle Y, Y \rangle$ is *bounded away from 0* in U_k , i.e., there exists a positive constant ν such that $-\langle Y, Y \rangle \geq \nu > 0$ in U_k ;
7. p and γ are not b-conjugate, i.e., for any brachistochrone σ of energy k in $\mathcal{B}_{p,\gamma}^{(1)}(k)$, the points $\sigma(0) = p$ and $\sigma(1)$ are not b-conjugate along σ .

We denote by $B_{p,\gamma}(k)$ the set of brachistochrones in $\mathcal{B}_{p,\gamma}^{(1)}(k)$; moreover, let $\mathcal{C}_{p,\gamma}^0$ denote the set of continuous paths joining p and γ in U_k :

$$\mathcal{C}_{p,\gamma}^0 = \left\{ z \in C^0([0, 1], U_k) : z(0) = p, z(1) \in \gamma(\mathbb{R}) \right\},$$

endowed with the topology of the uniform convergence.

The following is the main result of the Section:

Theorem 8.1. *Under the assumptions 1—7 above, given any coefficient field \mathcal{K} , the following equality between formal power series in the variable $\lambda \in \mathcal{K}$ holds true:*

$$(8.1) \quad \sum_{\sigma \in B_{p,\gamma}(k)} \lambda^{\mu(\sigma)} = \sum_{i=1}^{\infty} \dim(H_i(\mathcal{C}_{p,\gamma}^0, \mathcal{K})) \lambda^i + (1 + \lambda) Q(\lambda),$$

where $\mu(\sigma)$ is the geometric index of the brachistochrone σ , $H_i(\mathcal{C}_{p,\gamma}^0, \mathcal{K})$ is the i -th homology vector space of $\mathcal{C}_{p,\gamma}^0$ with coefficients in \mathcal{K} , and Q is a formal power series in λ with coefficients in $\mathbb{N} \cup \{+\infty\}$.

The Morse relations (8.1) can be used to derive a series of information about the number of brachistochrones joining p and γ and with a given energy value.

Remark 8.2. If the open set U_k is contractible, then also the space $\mathcal{C}_{p,\gamma}^0$ is contractible, and thus, for every field \mathcal{K} , its homology spaces $H_i(\mathcal{C}_{p,\gamma}^0, \mathcal{K})$ vanish for all $i > 0$ and $H_0(\mathcal{C}_{p,\gamma}^0, \mathcal{K}) \simeq \mathcal{K}$. In this case, under the assumptions 1—7 above, formula (8.1) becomes:

$$(8.2) \quad \sum_{\sigma \in B_{p,\gamma}(k)} \lambda^{\mu(\sigma)} = 1 + (1 + \lambda)Q(\lambda).$$

Setting $\lambda = 1$ in (8.2), we get immediately that the number of travel time brachistochrones of energy k between p and γ is either infinite (if $Q(1) = +\infty$) or *odd* (if $Q(1) < +\infty$).

On the other hand, if U_k is not contractible, then, since γ is contractible in U_k (because of the injectivity of γ), then there are infinitely many indices i such that $\dim(H_i(\mathcal{C}_{p,\gamma}^0, \mathcal{K})) > 0$ (see [25]). Hence, if U_k is not contractible, then there exist infinitely many brachistochrones of energy k between p and γ in U_k .

In order to prove Theorem 8.1, we will use the functional E_{ϕ_k} (defined in (4.7)) in the space $\Omega_{p,\gamma}^{(1)} = \Omega_{p,\gamma}^{(1)}(U_k)$, and ϕ_k is the function defined in (4.17).

Since U_k is open, we need to use a *penalization* argument, as follows. We define the function:

$$(8.3) \quad \Psi_k = \langle Y, Y \rangle + k^2;$$

It is $\partial U_k = \Psi_k^{-1}(0)$, moreover $\Psi_k(q) > 0$ if and only if $q \in U_k$. By assumption 4, the Riemannian gradient $\nabla^{(R)} \Psi_k$ does not vanish on ∂U_k , where $\nabla^{(R)}$ denotes the gradient with respect to the Riemannian metric (2.2).

We define a family χ_ε of real functions of class C^2 , for $\varepsilon > 0$:

$$(8.4) \quad \chi(s) = e^s - (1 + s + \frac{s^2}{2}), \quad \chi_\varepsilon(s) = \begin{cases} \chi(s - \frac{1}{\varepsilon}), & \text{if } s \geq \frac{1}{\varepsilon}; \\ 0, & \text{if } s < \frac{1}{\varepsilon}. \end{cases}$$

Finally, for all $\varepsilon \in]0, 1]$, we define the *penalized* functional:

$$(8.5) \quad E_\varepsilon(w) = E_{\phi_k}(w) + \int_0^1 \chi_\varepsilon \left(\frac{1}{\Psi_k(w)^2} \right) dt.$$

For all $\varepsilon > 0$, E_ε is a functional of class C^2 on $\Omega_{p,\gamma}^{(1)}$, which satisfies good *compactness properties*, as it will be discussed below.

By the completeness of \overline{U}_k , it is not too difficult to prove that, for every $c \in \mathbb{R}$, the sublevel E_ε^c :

$$E_\varepsilon^c = \left\{ w \in \Omega_{p,\gamma}^{(1)}(U_k) : E_\varepsilon(w) \leq c \right\}$$

is a complete metric subspace of $\Omega_{p,\gamma}^{(1)}(U_k)$, with respect to the metric induced by the Hilbert structure (2.7).

Moreover, using the same techniques employed in [7], one proves the following two facts:

- E_ε satisfies the *Palais-Smale condition* at every level $c \in \mathbb{R}$, i.e., every sequence $\{w_n\}$ in E_ε^c such that³ $dE_\varepsilon(w_n)$ tends to 0 as $n \rightarrow \infty$, has a convergent subsequence in E_ε^c ;
- for all $c \in \mathbb{R}$ there exists $\delta(c) > 0$ and $\varepsilon(c) \in]0, 1]$ such that, for all $\varepsilon \in]0, \varepsilon(c)]$ and for all critical point w_ε of E_ε in $\Omega_{p,\gamma}^{(1)}(U_k)$ with $E_\varepsilon(w_\varepsilon) \leq c$, then w_ε is also a critical point for E_{ϕ_k} , and the following inequality holds:

$$\Psi_k(w_\varepsilon(t)) \geq \delta(c), \quad \forall t \in [0, 1].$$

In particular, it follows that if c is a regular value for E_{ϕ_k} , i.e., if there are no critical point for E_{ϕ_k} in $E_{\phi_k}^{-1}(c)$, using (8.4) and (8.5) we obtain the existence of a number $\varepsilon'(c) \in]0, \varepsilon(c)]$ such that, for all $\varepsilon \in]0, \varepsilon'(c)]$, c is a regular value also for the functional E_ε , and a curve $w \in \Omega_{p,\gamma}^{(1)}$ is a critical point for E_ε if and only if it is a critical point for E_{ϕ_k} (with $E_\varepsilon(w) = E_{\phi_k}(w)$) and:

$$m(w, E_\varepsilon) = m(w, E_{\phi_k}),$$

where $m(z, G)$ denotes the *Morse Index* of the functional G at the critical point z .

Then, using assumption 7, for all $\varepsilon \in]0, \varepsilon'(c)]$, every critical point w of E_ε in E_ε^c is *nondegenerate*, which allows to obtain the Morse Relations in E_ε^c (see Ref. [17]):

Proposition 8.3. *If c is a regular value for E_{ϕ_k} , then there exists $\varepsilon'(c) \in]0, 1]$ such that, for every $\varepsilon \in]0, \varepsilon'(c)]$, we have:*

$$(8.6) \quad \sum_{w \in \mathcal{G}_{p,\gamma}^c} \lambda^{m(w, E_{\phi_k})} = \sum_{i=0}^{\infty} \dim(H_i(E_\varepsilon^c, \mathcal{K})) \lambda^i + (1 + \lambda) Q_c(\lambda),$$

where $\mathcal{G}_{p,\gamma}^c$ is the set of horizontal geodesics between p and γ with energy less than or equal to c :

$$\mathcal{G}_{p,\gamma}^c = \left\{ w \in \Omega_{p,\gamma}^{(1)}(U_k) : dE_{\phi_k}(w) = 0, E_{\phi_k}(w) \leq c \right\},$$

and $Q_c(\lambda)$ is a polynomial in the variable λ with coefficients in \mathbb{N} . □

We recall that, given a topological pair (A, B) , i.e., a topological space A and a subspace $B \subset A$ with the induced topology, we say that B is a *weak deformation retract* of A if there exists a continuous map $H : A \times [0, 1] \rightarrow A$ such that:

1. $H(\cdot, 0)$ is the identity map of A ;
2. $H(B, s) \subset B$ for all $s \in [0, 1]$;
3. $H(A, 1) \subset B$.

Given a topological pair (A, B) , we denote by $P_\lambda(A, B)$ the Poincaré series of (A, B) in the variable λ , which is given by:

$$P_\lambda(A, B; \mathcal{K}) = \sum_{i=0}^{\infty} \dim(H_i(A, B; \mathcal{K})) \lambda^i,$$

where $H_i(A, B; \mathcal{K})$ is the i -th relative homology space of (A, B) with coefficients in the field \mathcal{K} .

Now, for $\delta > 0$, we denote by $\Omega_{p,\gamma}^{(1)}(\delta)$ the set of curves in $\Omega_{p,\gamma}^{(1)}$ whose image stays at distance greater or equal to δ from ∂U_k :

$$\Omega_{p,\gamma}^{(1)}(\delta) = \left\{ w \in \Omega_{p,\gamma}^{(1)} : \Psi_k(w(t)) \geq \delta \quad \forall t \in [0, 1] \right\}.$$

Using the results of Ref. [5], we can prove that if c is a regular value of E_{ϕ_k} , there exists $\delta_0 = \delta_0(c) > 0$ and $\varepsilon_0 = \varepsilon_0(c)$ such that, for all $\delta \in]0, \delta_0]$ and for all $\varepsilon \in]0, \varepsilon_0]$, the following

³here, by convergence to 0, we mean that $\|dE_\varepsilon(w_n)\|$ goes to zero, where $\|\cdot\|$ is the operator norm in the dual space of $T_{w_n} \Omega_{p,\gamma}^{(1)}$.

two statements hold:

$$(8.7) \quad \Omega_{p,\gamma}^{(1)}(\delta) \cap E_{\phi_k}^c \text{ is a weak deformation retract of } E_{\phi_k}^c,$$

$$(8.8) \quad \Omega_{p,\gamma}^{(1)}(\delta) \cap E_\varepsilon^c \text{ is a weak deformation retract of } E_\varepsilon^c.$$

Observe that, if ε is sufficiently small, we have

$$\Omega_{p,\gamma}^{(1)}(\delta) \cap E_{\phi_k}^c = \Omega_{p,\gamma}^{(1)}(\delta) \cap E_\varepsilon^c.$$

Then, using standard techniques in Algebraic Topology, from (8.7) and (8.8) we deduce easily that, if c_1 and c_2 are critical values of E_{ϕ_k} , with $c_1 < c_2$, then there exists $\varepsilon_0 \in]0, 1]$ such that, for all $\varepsilon \in]0, \varepsilon_0]$, the following identities between Poincaré series hold:

- $P_\lambda(E_\varepsilon^{c_2}; \mathcal{K}) = P_\lambda(E_{\phi_k}^{c_2}; \mathcal{K});$
- $P_\lambda(E_\varepsilon^{c_2}, E_\varepsilon^{c_1}; \mathcal{K}) = P_\lambda(E_{\phi_k}^{c_2}, E_{\phi_k}^{c_1}; \mathcal{K}).$

Using the above identities and the same technique of [5, Theorem 1.6], one passes to the limit as $c \rightarrow +\infty$ in (8.6), obtaining the Morse relations for the functional E_{ϕ_k} in $\Omega_{p,\gamma}^{(1)}(U_k)$:

Theorem 8.4. *Under assumptions 1—7, for all coefficient field \mathcal{K} , we have:*

$$(8.9) \quad \sum_{w \in \mathcal{G}_{p,\gamma}} \lambda^{m(w, E_{\phi_k})} = \sum_{i=0}^{\infty} \dim(H_i(\Omega_{p,\gamma}^{(1)}(U_k); \mathcal{K})) \lambda^i + (1 + \lambda) Q(\lambda),$$

where $\mathcal{G}_{p,\gamma}$ is the set of all horizontal geodesics between p and γ :

$$\mathcal{G}_{p,\gamma} = \left\{ w \in \Omega_{p,\gamma}^{(1)} : dE_{\phi_k}(w) = 0 \right\},$$

and $Q(\lambda)$ is a formal power series in λ (depending on the choice of \mathcal{K}) with coefficients in $\mathbb{N} \cup \{+\infty\}$. \square

We are finally ready to prove Theorem 8.1:

Proof of Theorem 8.1. By Proposition 4.5 and Lemma 4.3 we see that $w \in \mathcal{G}_{p,\gamma}$ if and only if $w = \mathcal{D}(\sigma)$, where \mathcal{D} is the deformation map of (4.9) and σ is a travel time brachistochrone of energy k between p and γ . By the first part of Theorem 7.12, the hypothesis 7 implies that every $w \in \mathcal{G}_{p,\gamma}$ is a nondegenerate critical point of E_{ϕ_k} in $\Omega_{p,\gamma}^{(1)}(U_k)$. Moreover, by Theorem 6.8, we have $m(w, E_{\phi_k}) = \mu^{\{k\}}$, while, by Theorem 7.12, it is $\mu^{\{k\}}(w) = \mu(\sigma)$. Then, formula (8.9) can be written as:

$$\sum_{\sigma \in B_{p,\gamma}(k)} \lambda^{\mu(\sigma)} = \sum_{i=0}^{\infty} \dim(H_i(\Omega_{p,\gamma}^{(1)}(U_k); \mathcal{K})) \lambda^i + (1 + \lambda) Q(\lambda).$$

Finally, it is well known ([18, Theorem 17.1]) that $\Omega_{p,\gamma}^{(1)}(U_k)$ has the same homotopy type of $\mathcal{C}_{p,\gamma}^0(U_k)$, which concludes the proof. \square

APPENDIX A. AN EXPLICIT CALCULATION OF THE HESSIAN OF T ON $\mathcal{B}_{p,\gamma}^{(1)}(k)$

In this appendix we show how to compute explicitly the second variation of the functional T , or, recalling (5.5), equivalently, of the functional F .

To this aim, we fix a brachistochrone σ of energy k between p and γ , and we consider the corresponding Lagrange multipliers λ and μ , given by (3.13) and (3.18).

In the next Lemma it is shown how to compute the second variation on constrained critical points with the method of Lagrange multipliers. For simplicity, the result will be stated and proved only for Hilbertian manifolds.

Lemma A.1. *Let M be a Hilbert manifold and E be a Hilbert space. Let $f : M \rightarrow \mathbb{R}$ and $g : M \rightarrow E$ be smooth maps. Suppose that $0 \in E$ is a regular value for g , i.e., the differential $dg(x)$ is surjective for all $x \in g^{-1}(0)$, in such a way that $N = g^{-1}(0)$ is a smooth submanifold of M . Let $x_0 \in N$ be a critical point for the restriction $f|_N$ and let $\Lambda \in E^*$ be the (unique⁴) associated Lagrange multiplier, i.e., $d(f - \Lambda \circ g)(x_0) = 0$. Then, the Hessian of $f|_N$ at x_0 in $T_{x_0}N$ is given by the restriction of the Hessian of $(f - \Lambda \circ g)$ to $T_{x_0}N$:*

$$(A.1) \quad H^{f - \Lambda \circ g}(x_0)|_{T_{x_0}N \times T_{x_0}N} = H^{f|_N}(x_0).$$

Proof. Let $v \in T_{x_0}N$ and $y :]-\varepsilon, \varepsilon[\rightarrow N$ be a smooth curve such that $y(0) = x_0$ and $y'(0) = v$. By (5.2), we have:

$$(A.2) \quad H^{f - \Lambda \circ g}(x_0)[v, v] = \frac{d^2((f - \Lambda \circ g) \circ y)}{ds^2} \Big|_{s=0};$$

since $(g \circ y) \equiv 0$, then

$$(A.3) \quad \frac{d^2((f - \Lambda \circ g) \circ y)}{ds^2} \Big|_{s=0} = \frac{d^2(f \circ y)}{ds^2} \Big|_{s=0} = H^{f|_N}(x_0)[v, v],$$

which concludes the proof. \square

By Lemma A.1, the Hessian $H^F(\sigma)$ is given by the restriction of the Hessian $H^{F_{\lambda,\mu}}(\sigma)$ to the space $T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$, where $F_{\lambda,\mu} : \Omega_{p,\gamma}^{(1)} \rightarrow \mathbb{R}$ is the functional given by:

$$(A.4) \quad \begin{aligned} F_{\lambda,\mu}(\sigma) &= \int_0^1 \left[\frac{1}{2} \langle \dot{\sigma}, \dot{\sigma} \rangle - \lambda \langle \dot{\sigma}, Y \rangle - \mu \left(\langle \dot{\sigma}, Y \rangle^2 + k^2 \langle \dot{\sigma}, \dot{\sigma} \rangle \right) \right] dt = \\ &= \int_0^1 \left[\frac{1}{2} \langle \dot{\sigma}, \dot{\sigma} \rangle - \mu \left(\langle \dot{\sigma}, Y \rangle^2 + k^2 \langle \dot{\sigma}, \dot{\sigma} \rangle \right) \right] dt. \end{aligned}$$

In order to compute the second variation of $F_{\lambda,\mu}$, we will consider smooth variations in $\Omega_{p,\gamma}^{(1)}$ of a brachistochrone σ , defined as follows.

Definition A.2. Given a curve $z \in \Omega_{p,\gamma}^{(1)}$, by a *variation* of z we will mean a map $\eta :]-\varepsilon, \varepsilon[\times [0, 1] \rightarrow \mathcal{M}$ such that:

1. $\eta(s, \cdot) \in \Omega_{p,\gamma}^{(1)}$ for all $s \in]-\varepsilon, \varepsilon[$;
2. $\eta(0, \cdot) = z$;
3. the map $s \mapsto \eta(s, \cdot)$ is smooth from $]-\varepsilon, \varepsilon[$ to $\Omega_{p,\gamma}^{(1)}$.

Given a variation η of $z \in \Omega_{p,\gamma}^{(1)}$, for all s and t there exists the derivative $\frac{\partial \eta}{\partial s}(s, t) \in T_{\eta(s,t)} \mathcal{M}$; the vector field along z given by $V(t) = \frac{\partial \eta}{\partial s}(0, t)$ is called the *variational vector field* corresponding to the variation η .

In the rest of this section, given a variation $\eta(s, t) = \sigma_s(t)$ in $\Omega_{p,\gamma}^{(1)}$ of a curve $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$, we will denote by $\frac{D}{ds}$ and $\frac{D}{dt}$, the operations of covariant derivative of the Levi-Civita connection

⁴the Lagrange multiplier Λ is unique, because $dg(x_0)$ is surjective. The relation $d(f - \Lambda \circ g)(x_0) = 0$ defines Λ uniquely.

of g in the directions of $\frac{\partial \eta}{\partial s}$ and $\frac{\partial \eta}{\partial t}$ for vector fields along η ; the usual symbols $\frac{d}{ds}$ and $\frac{d}{dt}$ will denote the differentials with respect to s and t of functions along η .

Since the Lie bracket $[\frac{D}{ds}, \frac{D}{dt}]$ vanish and the Levi-Civita connection is torsion free, we have the following commutation relations involving the operators $\frac{D}{ds}$, $\frac{D}{dt}$, $\frac{d}{ds}$ and $\frac{d}{dt}$:

$$(A.5) \quad \frac{D}{ds} \frac{d}{dt} = \frac{D}{dt} \frac{d}{ds}, \quad \frac{D}{ds} \frac{D}{dt} = \frac{D}{dt} \frac{D}{ds} + R\left(\frac{D}{ds}, \frac{D}{dt}\right), \quad \frac{D}{dt} \frac{D}{ds} = \frac{D}{ds} \frac{D}{dt} + R\left(\frac{D}{dt}, \frac{D}{ds}\right);$$

where R is the curvature tensor of the Lorentzian metric g , defined in (2.1).

We have the following:

Proposition A.3. *Let $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ be a brachistochrone of energy k between p and γ of travel time \mathcal{T}_σ . Then, the Hessian $H^F(\sigma)$ of the action functional F (see (2.43)) at σ is given by the following formula:*

$$(A.6) \quad \begin{aligned} H^F(\sigma)[\zeta, \zeta] &= \int_0^1 \frac{\langle Y, Y \rangle}{k^2 + \langle Y, Y \rangle} \left[\langle \nabla_{\dot{\sigma}} \zeta, \nabla_{\dot{\sigma}} \zeta \rangle + \langle R(\zeta, \dot{\sigma}) \zeta, \dot{\sigma} \rangle \right] dt + \\ &+ 2k \mathcal{T}_\sigma \int_0^1 \frac{\langle \nabla_{\dot{\sigma}} \zeta, \nabla_{\dot{\sigma}} Y \rangle + \langle R(\zeta, \dot{\sigma}) \zeta, Y \rangle}{k^2 + \langle Y, Y \rangle} dt + \\ &+ \frac{\langle Y, Y \rangle}{k^2 + \langle Y, Y \rangle} a_\zeta^2 \langle \nabla_Y Y, \dot{\sigma} \rangle \Big|_{t=1}, \end{aligned}$$

for all $\zeta \in T_\sigma \mathcal{B}_{p,\gamma}^{(2)}(k)$, where a_ζ is defined by $\zeta(1) = a_\zeta \cdot Y(\sigma(1))$.

Proof. The computation is done by brute force, as follows.

Let $\sigma \in \mathcal{B}_{p,\gamma}^{(1)}(k)$ be a fixed brachistochrone. Observe that σ is smooth. Hence, by a density argument, it suffices to restrict our attention to smooth variations. Let $\zeta \in T_\sigma \mathcal{B}_{p,\gamma}^{(1)}(k)$ be a fixed smooth variational vector field and let σ_s , $s \in]-\varepsilon, \varepsilon[$, be a smooth variation⁵ of σ in $\Omega_{p,\gamma}^{(1)}$ corresponding to ζ . This means that $\sigma_s \in \Omega_{p,\gamma}^{(1)}$ for all s , $\sigma_0 = \sigma$, the map $(s, t) \mapsto \sigma_s(t) \in \mathcal{M}$ is smooth, and $\zeta = \frac{d}{ds} \Big|_{s=0} \sigma_s$.

We differentiate the expression $F_{\lambda,\mu}(\sigma_s)$ with respect to s twice, and we evaluate at $s = 0$. From (A.4), we have:

$$(A.7) \quad \begin{aligned} \frac{d^2}{ds^2} \Big|_{s=0} F_{\lambda,\mu}(\sigma_s) &= \int_0^1 (1 - 2\mu k^2) \left(\left\langle \frac{D}{ds} \frac{D}{dt} \frac{d}{ds} \sigma_s, \dot{\sigma} \right\rangle + \langle \nabla_{\dot{\sigma}} \zeta, \nabla_{\dot{\sigma}} \zeta \rangle \right) dt \\ &- 2 \int_0^1 \mu (\langle \nabla_{\dot{\sigma}} \zeta, Y \rangle - \langle \zeta, \nabla_{\dot{\sigma}} Y \rangle)^2 dt \\ &+ 2 \int_0^1 \mu k \mathcal{T}_\sigma \left(\left\langle \frac{D}{ds} \frac{D}{dt} \frac{d}{ds} \sigma_s, Y \right\rangle + \langle \nabla_{\dot{\sigma}} \zeta, \nabla_{\dot{\sigma}} Y \rangle \right) dt \\ &- 2 \int_0^1 \mu k \mathcal{T}_\sigma \left(\left\langle \frac{D}{ds} \frac{d}{ds} \sigma_s, \nabla_{\dot{\sigma}} Y \right\rangle + \langle \zeta, \frac{D}{ds} \frac{D}{dt} Y \rangle \right) dt. \end{aligned}$$

Since σ is a brachistochrone, by Corollary 2.4 the second integral in (A.7) vanishes:

$$(A.8) \quad \int_0^1 \mu (\langle \nabla_{\dot{\sigma}} \zeta, Y \rangle - \langle \zeta, \nabla_{\dot{\sigma}} Y \rangle)^2 dt = 0.$$

By (A.5), the last term in (A.7) can be written as:

$$(A.9) \quad \left\langle \zeta, \frac{D}{ds} \frac{D}{dt} Y \right\rangle \Big|_{s=0} = \langle \zeta, \nabla_{\dot{\sigma}} \nabla_{\dot{\sigma}} Y \rangle + \langle R(\zeta, \dot{\sigma}) Y, \zeta \rangle.$$

⁵the existence of such variations, at least in the smooth case, is easily proven using the exponential map and standard arguments in Riemannian manifolds.

We now consider the three terms in (A.7) that contain two derivatives with respect to s , and, using (A.5), we write them as follows:

$$\begin{aligned}
 (A.10) \quad & \int_0^1 (1 - 2\mu k^2) \left\langle \frac{D}{ds} \frac{D}{dt} \frac{d}{ds} \sigma_s, \dot{\sigma} \right\rangle dt + \\
 & + 2 \int_0^1 \mu k \mathcal{T}_\sigma \left(\left\langle \frac{D}{ds} \frac{D}{dt} \frac{d}{ds} \sigma_s, Y \right\rangle - \left\langle \frac{D}{ds} \frac{d}{ds} \sigma_s, \nabla_{\dot{\sigma}} Y \right\rangle \right) dt = \\
 & = \int_0^1 \left[(1 - 2\mu k^2) \left\langle R(\zeta, \dot{\sigma}) \zeta, \dot{\sigma} \right\rangle + 2\mu k \mathcal{T}_\sigma \left\langle R(\zeta, \dot{\sigma}) \zeta, Y \right\rangle \right] dt + \\
 & + \int_0^1 (1 - 2\mu k^2) \left\langle \frac{D}{dt} \frac{D}{ds} \frac{d}{ds} \sigma_s, \dot{\sigma} \right\rangle dt + \\
 & + 2k \mathcal{T}_\sigma \int_0^1 \mu \left(\left\langle \frac{D}{dt} \frac{D}{ds} \frac{d}{ds} \sigma_s, Y \right\rangle - \left\langle \frac{D}{ds} \frac{d}{ds} \sigma_s, \nabla_{\dot{\sigma}} Y \right\rangle \right) dt.
 \end{aligned}$$

Integration by parts in the last two integrals of (A.10) gives:

$$\begin{aligned}
 (A.11) \quad & \int_0^1 (1 - 2\mu k^2) \left\langle \frac{D}{dt} \frac{D}{ds} \frac{d}{ds} \sigma_s, \dot{\sigma} \right\rangle dt \\
 & + 2k \mathcal{T}_\sigma \int_0^1 \mu \left(\left\langle \frac{D}{dt} \frac{D}{ds} \frac{d}{ds} \sigma_s, Y \right\rangle - \left\langle \frac{D}{ds} \frac{d}{ds} \sigma_s, \nabla_{\dot{\sigma}} Y \right\rangle \right) dt = \\
 & = \int_0^1 (1 - 2\mu k^2) \left\langle \frac{D}{dt} \frac{D}{ds} \frac{d}{ds} \sigma_s, \dot{\sigma} \right\rangle dt \\
 & + 2k \mathcal{T}_\sigma \int_0^1 \mu \left(\frac{d}{dt} \left\langle \frac{D}{ds} \frac{d}{ds} \sigma_s, Y \right\rangle - 2 \left\langle \frac{D}{ds} \frac{d}{ds} \sigma_s, \nabla_{\dot{\sigma}} Y \right\rangle \right) dt = \\
 & = \left((1 - 2\mu k^2) \left\langle \frac{D}{ds} \frac{d}{ds} \sigma_s, \dot{\sigma} \right\rangle + 2\mu k \mathcal{T}_\sigma \left\langle \frac{D}{ds} \frac{d}{ds} \sigma_s, Y \right\rangle \right) \Big|_{t=0}^{t=1} \\
 & + \int_0^1 \left\langle \frac{D}{ds} \frac{d}{ds} \sigma_s, 2\mu' k^2 \dot{\sigma} - (1 - 2\mu k^2) \nabla_{\dot{\sigma}} \dot{\sigma} - 2k \mathcal{T}_\sigma \mu' Y - 4\mu k \mathcal{T}_\sigma Y \right\rangle dt \\
 & = \left\langle \frac{D}{ds} \frac{d}{ds} \sigma_s, (1 - 2\mu k^2) \dot{\sigma} + 2\mu k \mathcal{T}_\sigma Y \right\rangle \Big|_{t=1} = \\
 & = \left\langle \frac{D}{ds} \frac{d}{ds} \sigma_s, \frac{\langle Y, Y \rangle}{k^2 + \langle Y, Y \rangle} \dot{\sigma} + \frac{k \mathcal{T}_\sigma}{k^2 + \langle Y, Y \rangle} Y \right\rangle \Big|_{t=1},
 \end{aligned}$$

because, by (3.21), we have:

$$2\mu' k^2 \dot{\sigma} - (1 - 2\mu k^2) \nabla_{\dot{\sigma}} \dot{\sigma} - 2k \mathcal{T}_\sigma \mu' Y - 4\mu k \mathcal{T}_\sigma Y = 0,$$

and, since $\sigma_s(0) \equiv p$,

$$\frac{D}{ds} \frac{d}{ds} \sigma_s(0) = 0.$$

Here, we have used the equalities:

$$1 - 2\mu k^2 = \frac{\langle Y, Y \rangle}{k^2 + \langle Y, Y \rangle}, \quad \text{and} \quad \mu' = -\frac{\langle \nabla_{\dot{\sigma}} Y, Y \rangle}{(k^2 + \langle Y, Y \rangle)^2}.$$

Let now $\alpha(s)$ be defined by:

$$(A.12) \quad \sigma_s(1) = \gamma(\alpha(s)).$$

We have:

$$\zeta(1) = \alpha'(0) \cdot Y(\sigma(1)),$$

hence, multiplying by $Y(\sigma(1))$, we obtain

$$(A.13) \quad \alpha'(0) = \frac{\langle \zeta, Y \rangle}{\langle Y, Y \rangle} \Big|_{t=1}.$$

Moreover, from (A.12) we easily get:

$$(A.14) \quad \frac{D}{ds} \Big|_{s=0} \frac{d}{ds} [\sigma_s(1)] = \alpha'(0) \cdot \nabla_{\zeta(1)} Y + \alpha''(0) \cdot Y(\sigma(1)).$$

Substitution of (A.13) and (A.14) into (A.11) gives:

$$(A.15) \quad \begin{aligned} & \left\langle \frac{D}{ds} \frac{d}{ds} \sigma_s, \frac{\langle Y, Y \rangle}{k^2 + \langle Y, Y \rangle} \dot{\sigma} + \frac{k \mathcal{T}_\sigma}{k^2 + \langle Y, Y \rangle} Y \right\rangle \Big|_{t=1} = \\ & \left\langle \alpha'(0) \cdot \nabla_{\zeta(1)} Y + \alpha''(0) \cdot Y(\sigma(1)), \frac{\langle Y, Y \rangle}{k^2 + \langle Y, Y \rangle} \dot{\sigma} + \frac{k \mathcal{T}_\sigma}{k^2 + \langle Y, Y \rangle} Y \right\rangle = \\ & = \left\langle \frac{\langle \zeta, Y \rangle}{\langle Y, Y \rangle} \cdot \nabla_{\zeta(1)} Y, \frac{\langle Y, Y \rangle}{k^2 + \langle Y, Y \rangle} \dot{\sigma} + \frac{k \mathcal{T}_\sigma}{k^2 + \langle Y, Y \rangle} Y \right\rangle \Big|_{t=1}. \end{aligned}$$

In conclusion, we have proven the equality:

$$(A.16) \quad \begin{aligned} & \int_0^1 (1 - 2\mu k^2) \left\langle \frac{D}{ds} \frac{D}{dt} \frac{d}{ds} \sigma_s, \dot{\sigma} \right\rangle dt + \\ & + 2 \int_0^1 \mu k \mathcal{T}_\sigma \left(\left\langle \frac{D}{ds} \frac{D}{dt} \frac{d}{ds} \sigma_s, Y \right\rangle - \left\langle \frac{D}{ds} \frac{d}{ds} \sigma_s, \nabla_{\dot{\sigma}} Y \right\rangle \right) dt = \\ & = \int_0^1 \frac{\langle Y, Y \rangle}{k^2 + \langle Y, Y \rangle} \langle R(\zeta, \dot{\sigma}) \zeta, \dot{\sigma} \rangle dt + \\ & + \int_0^1 \frac{k \mathcal{T}_\sigma}{k^2 + \langle Y, Y \rangle} \langle R(\zeta, \dot{\sigma}) \zeta, Y \rangle dt + \\ & + \frac{\langle \zeta, Y \rangle}{\langle Y, Y \rangle (k^2 + \langle Y, Y \rangle)} \left(- \langle Y, Y \rangle \langle \zeta, \nabla_{\dot{\sigma}} Y \rangle + k \mathcal{T}_\sigma \langle Y, \nabla_{\zeta} Y \rangle \right) \Big|_{t=1}. \end{aligned}$$

Observe that, since $\zeta(1) = a_\zeta \cdot Y(\sigma(1))$, with

$$(A.17) \quad a_\zeta = \frac{\langle \zeta, Y \rangle}{\langle Y, Y \rangle} \Big|_{t=1},$$

then the boundary term in (A.16) vanishes:

$$(A.18) \quad \langle Y, \nabla_{\zeta} Y \rangle \Big|_{t=1} = 0.$$

Since $\langle \zeta, \nabla_{\zeta} Y \rangle \equiv 0$, then:

$$0 = \frac{d}{dt} \langle \zeta, \nabla_{\zeta} Y \rangle = \langle \nabla_{\dot{\sigma}} \zeta, \nabla_{\zeta} Y \rangle + \langle \zeta, \nabla_{\dot{\sigma}} \nabla_{\zeta} Y \rangle,$$

hence

$$(A.19) \quad -\langle \zeta, \nabla_{\dot{\sigma}} \nabla_{\zeta} Y \rangle = \langle \nabla_{\dot{\sigma}} \zeta, \nabla_{\zeta} Y \rangle.$$

Finally, by the anti-symmetry of the curvature tensor R , we have:

$$(A.20) \quad -\langle R(\zeta, \dot{\sigma}) Y, \zeta \rangle = \langle R(\zeta, \dot{\sigma}) \zeta, Y \rangle.$$

Formula (A.6) now follows from (A.7), (A.8), (A.9), (A.16), (A.17), (A.18), (A.19) and (A.20). \square

Let's assume now that γ has no self intersection, which in particular implies that $\gamma(\mathbb{R})$ is an embedded submanifold of \mathcal{M} .

Remark A.4. *If we consider the submanifold $\Sigma = \gamma(\mathbb{R})$, then the second fundamental form S^γ takes the following form. For $q = \gamma(s_0)$ and $v_i = \nu_i \cdot Y(q)$, $i = 1, 2$, given a vector $n \in T_q \mathcal{M}$ which is orthogonal to $Y(q)$, we have:*

$$S_n^\gamma(v_1, v_2) = \nu_1 \nu_2 \cdot \langle \nabla_Y Y \mid_q, n \rangle.$$

This formula resembles the factor $a_\zeta^2 \cdot \langle \nabla_Y Y, \dot{\sigma} \rangle \big|_{t=1}$ that appears in the boundary term of $H^F(\sigma)[\zeta, \zeta]$ in formula (A.6). The reader should observe, though, that the vector $\dot{\sigma}(1)$ is not orthogonal to $Y(\sigma(1))$, because $\langle \dot{\sigma}, Y \rangle \equiv -k\mathcal{T}_\sigma \neq 0$.

APPENDIX B. F DOES NOT SATISFY THE PALAIS–SMALE CONDITION IN $\mathcal{B}_{p,\gamma}^{(2)}(k)$

We discuss a very simple example to prove that, in general, the travel time functional T , or the action functional F do not satisfy the Palais–Smale compactness condition in $\mathcal{B}_{p,\gamma}^{(2)}(k)$.

Let's consider $\mathcal{M} = \mathbb{R}^3$ to be the flat 3-dimensional Minkowski spacetime, with metric $\langle \cdot, \cdot \rangle$ given by $dx^2 + dy^2 - dz^2$ and $Y = \frac{\partial}{\partial z}$ the timelike Killing vector field on \mathcal{M} . Let $\langle \cdot, \cdot \rangle_o$ denote the Euclidean metric $dx^2 + dy^2$ in \mathbb{R}^2 .

We fix a point $p = (p_0, 0)$ in \mathcal{M} and a curve $\gamma(r) = (p_1, r)$, where $x_0, x_1 \in \mathbb{R}^2$, and a real constant $k > -\langle Y, Y \rangle \equiv 1$.

In this case, the set $\mathcal{B}_{p,\gamma}^{(2)}(k)$ consists of curves $\sigma(t) = (x(t), y(t), z(t))$ where $\mathbf{x}(t) = (x(t), y(t))$ is in $H^2([0, 1], \mathbb{R}^2)$ is a curve in \mathbb{R}^2 that joins p_0 and p_1 , $z \in H^2([0, 1], \mathbb{R})$, $z(0) = 0$, and there exists a positive constant \mathcal{T}_σ such that:

$$\dot{z} = k\mathcal{T}_\sigma, \quad \langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle_o - \dot{z}^2 = -\mathcal{T}_\sigma^2,$$

and so:

$$\langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle_o = (k^2 - 1)\mathcal{T}_\sigma^2 > 0 \text{ (constant)}.$$

It is easy to see that the map $(\mathbf{x}, z) \mapsto \mathbf{x}$ gives a diffeomorphism of $\mathcal{B}_{p,\gamma}^{(2)}(k)$ and the Hilbert manifold:

$$(B.1) \quad \Omega_c^{(2)}(p_0, p_1) = \left\{ \mathbf{x} \in H^2([0, 1], \mathbb{R}^2) : \mathbf{x}(0) = p_0, \mathbf{x}(1) = p_1, \langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle_o \equiv C_{\mathbf{x}} = \text{const.} > 0 \right\};$$

moreover, the travel time functional T and the action functional F on $\mathcal{B}_{p,\gamma}^{(2)}(k)$ are transformed respectively into (constant multiples of) the Euclidean length functional L and the Euclidean energy functional E on $\Omega_c^{(2)}(p_0, p_1)$:

$$L(\mathbf{x}) = \int_0^1 \langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle_o dt, \quad E(\mathbf{x}) = \frac{1}{2} \int_0^1 \langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle_o^2 dt.$$

It is not hard to prove that the only critical point of L and E on $\Omega_c^{(2)}(p_0, p_1)$ is the Euclidean geodesic, i.e., the straight segment, between p_0 and p_1 in \mathbb{R}^2 .

On the other hand, if $p_0 \neq p_1$, the manifold $\Omega_c^{(2)}(p_0, p_1)$ is complete, and it is easy to see that its first homotopy group is infinite. Thus, if either L or E satisfied the Palais–Smale condition on $\Omega_c^{(2)}(p_0, p_1)$, by standard techniques of Critical Point Theory one could prove the existence of infinitely many distinct geodesics between p_0 and p_1 in \mathbb{R}^2 , which is clearly absurd.

It follows that neither T nor F satisfies the Palais–Smale condition on $\mathcal{B}_{p,\gamma}^{(2)}(k)$. The same argument shows that neither T nor F satisfies the Palais–Smale condition in any set of curves satisfying a regularity that implies the C^1 -regularity.

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